MOTION OF EXTENDED OBJECTS IN GRAVITATIONAL FIELD WITH TORSION

Marko Vojinović

PhD thesis, Belgrade 2008.

Contents

Abstract

In this paper, we consider the motion of extended objects in gravitational field with torsion. Using multipole formalism, from the assumptions of Poincaré invariance and localization of matter on some hypersurface, we derive equations of motion for a p-brane, along with appropriate boundary conditions. Then we analyze the most important cases of particle and string, thereby obtaining some interesting consequences. Namely, it turns out that the spin of Dirac particle does not couple to spacetime curvature, and furthermore can be neglected in single-pole approximation. Also, we considered one gross model of a meson which consists of a Nambu-Goto string with two particles attached to its ends. The model allows for the derivation of Regge trajectories with a correction due to internal angular momenta of the particles at the string ends.

Foreword

Determining of the trajectory of bodies moving through space is one of the oldest problems for physics. Since the time of Newton, one introduces a notion of point particle, then postulates the motion of the *free* particle (first Newton's law), and postulates the equation of dynamics which governs the particle motion under the influence of external forces (second Newton's law). A theory dealing with the motion of particles is usually called *mechanics*. There are, however, phenomena in Nature which cannot be successfully described with a model of particles interacting upon each other with forces. Because of that, a new concept of description has been introduced, where the central object under consideration is the *field*, an entity that exists in all points of spacetime and in each of these points has a certain "value". All physical phenomena are then described as consequences of the properties and dynamics of a field (one or more), and one postulates new *field equations*, which describe the dynamics of each field alone and their interactions. Theories built in this way are usually called *field theories*.

Although as a method a field theory is very successful in describing natural phenomena, it turns out that it is nontrivial to describe motion of particles within its framework, i.e. to derive laws of mechanics out of field equations. Namely, a field is based on a concept of *continuum*, and it is not obvious how from this continuum one should build discrete, *pointlike* particles that we see in experiments. This problem can be partially circumvented by introducing a "hybrid" theory that mixes mechanics with a field theory, containing both fields with their postulates and particles with their. In the process, interaction of the field with the particle is also postulated, based on experimental data. This kind of construction is not satisfactory from a conceptual point of view, because one would expect particles not to be introduced separately, but as some localized configurations of the fields themselves, the so-called "kinks". In such setting field equations would in the end determine the dynamics of these kinks, which would open a possibility to avoid postulating the motion of particles, but rather to derive it from the all-encompassing field equations.

The need for this kind of approach emerged with the appearance of general theory of relativity, which describes gravitation as curvature of spacetime continuum. Namely, question of particle motion in an external gravitational field became nontrivial, because the notion of "uniform straight line motion of free particle" became undefined since there are no straight lines in curved spacetime. This problem could be in principle addressed via generalization of the "straight line" notion to a geodesic. But this approach has limitations, since it can be successfully applied only to Riemann geometry, where such generalization is unique. Already when one considers more general geometries like Riemann-Cartan, one can find *two* characteristic lines — autoparallel and extremal — which can both be considered good candidates for generalization of the straight line. In such a situation there is ambiguity in the choice of the postulate of particle motion.

On the other hand, description of particles as kinks in a field theory, besides being aesthetically more pleasing, does not suffer of these ambiguities. Incidentally, the kink approach should provide for a method of deriving effective equations of kink motion. This problem was addressed in their time by Einstein, Infeld, Hoffmann, Mathisson, Papapetrou and others [1, 2, 3, 4, 5, 6, 7]. A consistent, partly systematic and of course successful solution for the case of Riemann geometry is due to Papapetrou [4], which was subsequently generalized to the geometries with torsion [8, 9, 10, 11, 12]. Nevertheless, these methods suffered from the inability to describe the Dirac particle [11, 12], and were not manifestly covariant while geometric interpretation was quite blur. This situation rendered the method unfeasible for application to kinks with structure other than pointlike but rather extended — first of all the string, and then branes in general.

Motivation for the description of strings and branes as kinks in a field theory emerged from yet another area of physics — the string theory. String theory was initially conceived as an approach to explain meson resonances in physics of strong interactions, and later in attempts for unification of all interactions and formulation of "theory of everything". As one of the first steps in the construction of string theory, equations of motion of free relativistic bosonic string were postulated [13, 14]. After that (both classical and quantum) string field theory has been constructed, and within its framework one addressed the problem of motion of a single string in an external field of all other strings [15, 16, 17, 18]. In a theory constructed in this way, one encounters an effective symmetric field $\mu\nu$, Kalb-Ramond antisymmetric field $B_{\mu\nu}$ and the dilaton scalar field Φ, which interact with the string in question. Analysis of equations of motion and interaction of the string with these fields indicated that the field $g_{\mu\nu}$ has something to do with spacetime curvature, field $B_{\mu\nu}$ with spacetime torsion, while the dilaton field Φ is connected to spacetime nonmetricity [19, 20, 21, 22, 23]. However, equations of motion for the string have been essentially postulated, based on some more or less plausible arguments.

This situation indicated the need of a kink-like approach to the problem of string motion within a field theory framework, namely to generalize the Papapetrou method from particles to extended objects and employ it to derive effective equations of motion for a general kink with the p-brane shape. Such generalization is nontrivial but possible [24, 25, 26], and represents the central topic of this paper. The problem is solved for the general case of a p-brane moving in a D-dimensional spacetime with curvature and torsion. Of course, in order to clarify the geometric picture and derive the equations of motion in a manifestly covariant way, it was necessary to invent a completely new mathematical formalism which could be employed for the purpose of generalizing to the case of extended objects. This new "language" is called *multipole formalism*, and we dedicate the first part of this paper to its founding. The second part represents the application of multipole formalism for derivation of desired equations of motion, while the third part deals with various examples. One of interesting aspects of geometrically clear picture of derivation, which actually enforced itself upon the authors, is consisted of successful description of the Dirac particle case, and some attention will be dedicated to this as well.

The paper is divided in three chapters. First chapter deals with the formalism of multipole approximations in spaces with curvature and torsion. After a brief recapitulation of Riemann-Cartan geometry in the first section [27], in the second section we introduce the concept of expanding a function into the series of derivatives of Dirac δ function, first on an elementary level, and subsequently in the general case of space with curvature and torsion. Third section is devoted to casting of the δ series into a manifestly covariant form, which is very important for later analysis. After that, in fourth section we introduce the concept of multipole approximations via truncation of the δ series at some point. As the most important special cases, we single out the so called *single-pole* and *pole-dipole* approximations to be used later on. The manifest covariance of the δ series guarantees the truncation to be fully covariant, and thus multipole approximations properly defined. Fifth section deals with the analysis of symmetries of the pole-dipole approximation. In addition to spacetime and worldsheet diffeomorphisms, we discover two extra symmetries. The first of these represents the fact that the manifestly covariant notation there exist nonphysical variables, while the second is a consequence of the arbitrariness in choice of the hypersurface around which the δ series is being expanded. The analysis of these symmetries concludes the general analysis of multipole formalism, and also the first chapter.

The second chapter deals with the application of the δ series in derivation of effective equations of motion for a p-brane. In the first section we discuss Poincaré invariance of the matter Lagrangian, based on which one obtains the covariant conservation laws of matter stress–energy and spin tensors. It turns out that that the antisymmetric components of stress–energy tensor are not independent variables and that they can be completely eliminated in favor of the symmetric components and spin tensor components, which leaves us with only one covariant conservation law that connects them. In the second section we introduce the assumption of matter localization along some hypersurface, and in pole-dipole approximation we derive effective equations of motion and boundary conditions for a p-brane made of scalar matter. Equations of motion are derived in two main steps. First one substitutes the pole-dipole δ expansion of stress–energy tensor into the covariant conservation law. After appropriate casting into the suitable form, covariant conservation law breaks into three coupled equations and three boundary conditions. The second step represents decoupling, i.e. diagonalization of that system of equations, which is performed by solving all algebraic equations and introducing new, more suitable variables. As a result, we end up with an equation of motion for a p-brane, equation for the precession of the internal angular momentum and appropriate boundary conditions. In these equations we are left with a number of free parameters, which are given physical interpretation in the third section. They are recognized as the effective $(p+1)$-dimensional stress–energy tensor of the p-brane m^{ab}, p-dimensional stress–energy tensor of its boundary N^{ij}, and the internal angular momentum current $L^{\mu\nu a}$. These variables actually determine the structure and properties of matter the p-brane is made of. Armed with this knowledge, in fourth section we concentrate on the derivation of equations of motion for matter with nonzero spin in Riemann-Cartan spacetime. First we formulate the pole-dipole approximation for the stress–energy and spin tensors, and then we derive the equation of motion for a p-brane, an equation of angular momentum precession and appropriate boundary conditions, similar in procedure to previous two sections. These equations represent the main result of the paper. Along with other so far discussed free parameters, in these equations appears the current of the spin angular momentum, and they are all coupled not only to external curvature, but also to external torsion field. The section ends with the discussion of the very important case of single-pole approximation, where orbital angular momentum is completely absent, while the spin angular momentum remains present. In the process some restrictions on spin currents arise and reduce the number of free parameters in the theory. These will play a very important role in the analysis of Dirac matter.

Third chapter is devoted to examples. In the first section we discuss the case of 0-brane, i.e. the particle. We analyze the motion of the particle in pole-dipole approximation and demonstrate the connection between the second extra symmetry and the choice of center-of-mass line. Then we turn to the motion of the particle with nonzero spin in the single-pole approximation, and discuss the very important special case of Dirac point particle, i.e. matter with totally antisymmetric spin tensor. It turns out that the interaction of spin with spacetime curvature disappears, while all but the axial component of the torsion also decouple. The precession equation is reduced to an algebraic equation for the spin and contorsion, which represents a very strange result and suggests that maybe the spin of the Dirac particle is negligible in single-pole approximation. In order to examine this possibility, we construct one concrete model for the Dirac particle as a wave-packet in Minkowski spacetime, and then in the single-pole limit we demonstrate that the spin can really be considered negligible. As a result of this it turns out that the Dirac point particle travels along a geodesic line just as a scalar particle, without coupling to spacetime torsion. Based on this analysis, we formulate a criterion to check if the spin of the given matter configuration can be considered negligible in single-pole approximation. This ends the analysis of the particle case. The second section deals with the case of 1-brane, i.e. the string. First we demonstrate that the equations of motion contain the special case of Nambu-Goto string, along with Neumann boundary conditions. Since it has no spin, the Nambu-Goto string does not feel the presence of torsion, while it feels the presence of curvature only through Christoffel connection, so the equations of motion turn out to be familiar extremal surface equations. The next short example deals with a massive rod at rest while rotating around its longitudinal axis. It is demonstrated that equations of motion are satisfied if effective stress–energy tensor and angular momentum of the rod are conserved, while there is no energy nor angular momentum flow through the rod endpoints. Next, we analyze one more complicated system consisted of a Nambu-Goto string with two massive particles with nonzero internal angular momentum attached to its ends. Such a system represents one elementary model of a meson, where quarks are approximated with two rotating particles, while the gluon field connecting them is approximated with a Nambu-Goto string. Equations of motion are explicitly solved for one concrete configuration where the string lies along a straight line and together with the particles rotates uniformly around its center. From the expression for total energy and total angular momentum of this system one derives the law of Regge trajectories, along with a correction coming from internal angular momenta of two particles. Finally, we analyze the single-pole case with the Dirac particles at the ends, where this correction is seen to vanish due to negligibility of spin in this approximation.

At the end we give final remarks, recapitulate the results and discuss possible topics for further research.

I wish to express my gratitude to my mentor, Prof. dr Milovan Vasilić, for the guidance and help in all aspects of writing this paper, from the choice of the theme over considerations of conceptual issues to advice of technical character.

The conventions of the paper are the following. Greek indices from the middle of the alphabet, $\mu, \nu, \dots$, are spacetime indices and take values $0, 1, \dots, D-1$. Greek beginning-alphabet indices, $\alpha, \beta, \dots$, take only spacelike values $1, 2, \dots, D-1$. Latin beginning-alphabet indices, $a, b, c, \dots$, are worldsheet indices of the p-brane, and take values $0, 1, \dots, p$. Latin middle-alphabet indices, $i, j, k, \dots$, are the indices of the worldsheet boundary and take values $0, 1, \dots, p-1$. Spacetime, worldsheet and boundary coordinates are denoted respectively as x^μ, ξ^a and λ^i, while appropriate metric tensors are denoted as $g_{\mu\nu}(x)$, $\gamma_{ab}(\xi)$ and $h_{ij}(\lambda)$. Their signatures are $\mathrm{diag}[-1, +1, \dots, +1]$. We shall be mainly interested in the physically relevant case of $D = 4$ spacetime dimensions.

Chapter 1

MULTIPOLE FORMALISM

Multipole formalism basically represents the description of a given function using its *multipoles*, which appear as coefficients in the series of *derivatives of the Dirac δ function*. This chapter is dedicated to the formulation and analysis of some general properties of the δ series, with the emphasis on the case where the target space has Riemann-Cartan structure. This is why we start this chapter with a short recapitulation of geometric properties of Riemann-Cartan spaces. After that we shall deal with the definition and basic properties of the δ series, define multipole approximations and consider symmetries that the multipoles exhibit.

1.1 Riemann-Cartan geometry

Let us begin with a short recapitulation of the most important notions in Riemann-Cartan geometry [27]. This geometry is the one where one independently defines the rule for calculating the distance between points and the rule for parallel transport of a vector from one point to another. Information about those two operations is encoded in the metric tensor, $g_{\mu\nu}(x)$, and connection coefficients, $\Gamma^{\lambda}{}_{\mu\nu}(x)$. The metric tensor enters in the expression for the "squared line element",

$$ds^2 = g_{\mu\nu}(x)dx^{\mu}dx^{\nu},$$

while connection coefficients enter the definition of covariant derivative

$$D_{\nu}v^{\mu} \equiv \partial_{\nu}v^{\mu} + \Gamma^{\mu}{}_{\lambda\nu}v^{\lambda}$$

and are not necessarily symmetric with respect to two lower indices. In addition to this, in Riemann-Cartan geometry one enforces the metricity condition

$$D_{\lambda}g_{\mu\nu} = 0,$$

which implies that the connection can be uniquely split into the Levi-Civita connection and the contorsion tensor:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{{}^{\lambda}_{\mu\nu}\right\} + K^{\lambda}{}_{\mu\nu}, \qquad \left\{{}^{\lambda}_{\mu\nu}\right\} \equiv \frac{1}{2}g^{\lambda\rho}\left(\partial_{\mu}g_{\rho\nu} + \partial_{\nu}g_{\rho\mu} - \partial_{\rho}g_{\mu\nu}\right). \tag{1.1}$$

Basic geometric properties of these kind of spaces are characterized with curvature and torsion tensors:

$$\mathcal{R}^{\mu}{}_{\nu\lambda\rho} \equiv \partial_{\lambda}\Gamma^{\mu}{}_{\nu\rho} - \partial_{\rho}\Gamma^{\mu}{}_{\nu\lambda} + \Gamma^{\mu}{}_{\sigma\lambda}\Gamma^{\sigma}{}_{\nu\rho} - \Gamma^{\mu}{}_{\sigma\rho}\Gamma^{\sigma}{}_{\nu\lambda}, \qquad T^{\lambda}{}_{\mu\nu} \equiv \Gamma^{\lambda}{}_{\nu\mu} - \Gamma^{\lambda}{}_{\mu\nu}.$$

These satisfy two Bianchi identities,

$$D_{\mu}T^{\lambda}{}_{\nu\sigma} + T^{\lambda}{}_{\rho\sigma}T^{\rho}{}_{\mu\nu} - \mathcal{R}^{\lambda}{}_{\mu\nu\sigma} + \mathrm{cp}(\mu\nu\sigma) = 0,$$

$$D_\lambda \mathcal{R}^{\rho\tau}{}_{\mu\nu} + T^\sigma{}_{\lambda\mu} \mathcal{R}^{\rho\tau}{}_{\sigma\nu} + \mathrm{cp}(\lambda\mu\nu) = 0,$$

where cp() denotes cyclic permutations of appropriate indices. There is a bijection between the torsion and contorsion tensors,

$$T^\lambda{}_{\mu\nu} = K^\lambda{}_{\nu\mu} - K^\lambda{}_{\mu\nu}, \qquad K^\lambda{}_{\mu\nu} = -\frac{1}{2}\left(T^\lambda{}_{\mu\nu} + T_{\nu\mu}{}^\lambda + T_{\mu\nu}{}^\lambda\right),$$

so it does not matter which of the two we use, since they both contain the same information.

In addition to these tensors, it is useful to introduce also the Riemann covariant derivative and the Riemann curvature tensor:

$$\nabla_\mu \equiv D_\mu \Big|_{\Gamma \to \{\}}, \qquad R^\mu{}_{\nu\lambda\rho} \equiv \mathcal{R}^\mu{}_{\nu\lambda\rho}\Big|_{\Gamma \to \{\}}.$$

Clearly, the Riemann derivative also satisfies the metricity condition, $\nabla_\lambda g_{\mu\nu} = 0$. Since there is a relation between the full connection $\Gamma^\lambda{}_{\mu\nu}$ and the Levi-Civita connection $\left\{{\lambda \atop \mu\nu}\right\}$, there is an appropriate relation between the two curvature tensors,

$$\mathcal{R}^\mu{}_{\nu\lambda\rho} = R^\mu{}_{\nu\lambda\rho} + 2\nabla_{[\lambda} K^\mu{}_{\nu\rho]} + 2K^\mu{}_{\sigma[\lambda} K^\sigma{}_{\nu\rho]},$$

where the indices in square brackets are assumed to be antisymmetrized. Also, the Riemann curvature tensor satisfies appropriate Bianchi identities:

$$R^\lambda{}_{\mu\nu\sigma} + \mathrm{cp}(\mu\nu\sigma) = 0, \qquad \nabla_\lambda R^{\rho\sigma}{}_{\mu\nu} + \mathrm{cp}(\lambda\mu\nu) = 0.$$

After we have introduced all important geometric notions and fixed the conventions, we turn to the multipole formalism in this type of geometry. The first step represents the expansion in the series of derivatives of Dirac δ function.

1.2 Series of derivatives of δ function

Let us discuss first the very idea of the δ series in its most basic one-dimensional case. Let $V : \mathbb{R} \to \mathbb{R}$ be some given function, and let $z \in \mathbb{R}$ be some point. Then, if for every $n \in \mathbb{N}_0$ there exists an integral

$$b_n \equiv \frac{(-1)^n}{n!} \int_{\mathbb{R}} dx\, (x-z)^n V(x)$$

(the so-called n-th moment of the function V), one can define the expansion of the function V into series of derivatives of the δ function around the point z, as:

$$V(x) = \sum_{n\in\mathbb{N}_0} b_n \frac{d^n}{dx^n} \delta(x-z). \tag{1.2}$$

The equality can be easily checked by multiplying with some power of x and integrating term by term.

Consider a simple example. As a function V choose a Gaussian function, $V(x) = e^{-x^2}$, and expand it into the δ series around the point $z = 0$. Calculating the coefficients b_n and substituting them into (1.2) we obtain:

$$e^{-x^2} = \sum_{n\in\mathbb{N}_0} \frac{\sqrt{\pi}}{4^n n!} \frac{d^{2n}}{dx^{2n}} \delta(x) = \sqrt{\pi}\delta(x) + \frac{\sqrt{\pi}}{4}\delta''(x) + \frac{\sqrt{\pi}}{32}\delta''''(x) + \frac{\sqrt{\pi}}{384}\delta^{(6)}(x) + \dots$$

We see that in the series only even derivatives of the δ function appear, since the Gaussian is an even function, and also that every coefficient multiplying the higher derivative is smaller than previous ones. This is a consequence of the fact that the Gaussian is "localized" precisely around the point $z = 0$ with respect to which we chose to perform the expansion. If we had chosen some other point, the coefficients would be different and would not form a descending series. This property is clear also from a geometrical standpoint, because if we observe a Gaussian "from a distance", it will be more and more similar to the single δ function precisely at the point $z = 0$.

These results can be generalized to the case of D-dimensional Riemann-Cartan spacetime. Define some $(p+1)$-dimensional hypersurface $\mathcal{M}$ in spacetime, described via parametric equations $x^\mu = z^\mu(\xi)$, where ξ^a are parameters, i.e. coordinates on the hypersurface. Assume that the hypersurface is everywhere regular and that the coordinates ξ^a well defined. Also, the surface boundary $\partial\mathcal{M}$ is described via parametric equations $\xi^a = \zeta^a(\lambda)$, where λ^i are coordinates at the boundary. Index a takes values $0, \dots, p$, while i takes values $0, \dots, p-1$. In addition, we shall be interested only in time-unbounded surfaces, i.e. the ones that have a nontrivial intersection with any and every space section of spacetime. Given that, introduce the tangent vectors on the surface $\mathcal{M}$ and the induced metric tensor:

$$u^\mu_a \equiv \frac{\partial z^\mu}{\partial \xi^a}, \qquad \gamma_{ab} \equiv g_{\mu\nu}(z) u^\mu_a u^\nu_b.$$

The induced metric is assumed to be nondegenerate, $\gamma \equiv \det(\gamma_{ab}) \neq 0$, and has Minkowski signature. Also, introduce the tangent vectors and induced metric on the boundary $\partial\mathcal{M}$:

$$v^a_i \equiv \frac{\partial \zeta^a}{\partial \lambda^i}, \qquad h_{ij} \equiv \gamma_{ab}(\zeta) v^a_i v^b_j,$$

as well as a unit vector n_a orthogonal to the boundary $\partial\mathcal{M}$ which lies in $\mathcal{M}$:

$$n_a n^a = 1, \qquad n_a v^a_i \equiv 0.$$

Vectors v^a_i and n_a are also spacetime vectors, so we have the relations $v^\mu_i = v^a_i u^\mu_a$ and $n^\mu = n^a u^\mu_a$.

Now consider some tensor field $V^{\mu\nu}(x)$, and define its δ series expansion around the surface $\mathcal{M}$ as:

$$V^{\mu\nu}(x) = \int_{\mathcal{M}} d^{p+1}\xi \sqrt{-\gamma} \left[b^{\mu\nu}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + b^{\mu\nu\rho}(\xi) \nabla_\rho \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + \dots \right], \tag{1.3}$$

where $g \equiv \det(g_{\mu\nu})$ is the determinant of the spacetime metric, while the derivatives are with respect to x. For concreteness we have chosen the field V to be a second rank tensor, while all other cases are analogously treated. Regarding this, note that the b coefficients carry all indices of the field V, and in addition have extra indices contracted to covariant derivatives.

The series is defined for those fields V for which all moments are finite, i.e. those that are different from zero in vicinity of some surface and exponentially falling to zero as one moves away from it. If we choose precisely this surface (or some nearby one) for the expansion into δ series, it is natural to assume that every next coefficient in the series will be smaller than the previous, as in the Gaussian example. According to this we introduce the "degree of smallness" of each coefficient:

$$b \sim \mathcal{O}_0, \qquad b^\rho \sim \mathcal{O}_1, \qquad b^{\rho\sigma} \sim \mathcal{O}_2, \qquad \dots, \tag{1.4}$$

and define $\mathcal{O}_m \mathcal{O}_n = \mathcal{O}_{m+n}$ in order to be able to quantify the smallness of the product of variables. This formalism will later turn out to be very useful.

In regard to the δ series as defined above, it is necessary to comment on the question of the choice of ∇_μ over D_μ in the definition. It turns out that the two choices are equivalent, up to a redefinition of the b coefficients. Indeed, we see that according to relation (1.1), we have for example for a quadruple term:

$$\begin{aligned} b^{\mu\nu\rho\sigma} \nabla_\rho \nabla_\sigma \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} &= b^{\mu\nu\rho\sigma} D_\rho D_\sigma \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + \\ &+ b^{\mu\nu\rho\sigma} K^\lambda{}_{\sigma\rho} D_\lambda \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} - b^{\mu\nu\rho\sigma} \left(D_\lambda K^\lambda{}_{\sigma\rho} \right) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \end{aligned}$$

(here the contorsion and its derivative are evaluated at the point $x = z$), so a simple regrouping and redefinition of the b coefficients allows the same δ series to be rewritten using D_μ. All things equal, the Riemann derivative is simpler to calculate with, so we prefer it and adopt it in the definition of the series.

1.3 Proof of covariance

Given the expansion (1.3), one can ask the question of transformation law for the b coefficients under the spacetime diffeomorphisms, i.e. transformations of coordinates $x \to x' = x'(x)$. It is not obvious that b's transform as tensors, and a careful check can even show that they do not. A natural question arises — is it possible to rewrite the series in a manifestly covariant manner? The answer is affirmative, and to demonstrate this, first define the functional

$$V[f] \equiv \int d^D x \sqrt{-g} V^{\mu\nu}(x) f_{\mu\nu}(x). \tag{1.5}$$

Here $f_{\mu\nu}(x)$ is a test-function which is assumed to have compact support[1], so we may commute the spacetime integrals with integrals over the surface $\mathcal{M}$. Then, after inserting (1.3) and integrating term by term over spacetime, we write the functional in the form:

$$V[f] = \int_{\mathcal{M}} d^{p+1}\xi \sqrt{-\gamma} \Big[I_0(b_0 f) + I_1(b_1 f) + \dots \Big], \tag{1.6}$$

where

$$I_n(b_n f) \equiv \int d^D x \sqrt{-g}\, b^{\mu\nu\rho_1\dots\rho_n} f_{\mu\nu}(x) \nabla_{\rho_1} \dots \nabla_{\rho_n} \frac{\delta^{(D)}(x-z)}{\sqrt{-g}}.$$

In this expression we perform a series of partial integrations, which brings us to

$$I_n(b_n f) = (-1)^n \nabla_{\rho_n} \dots \nabla_{\rho_1} \Big[b^{\mu\nu\rho_1\dots\rho_n} f_{\mu\nu}(x) \Big] \Big|_{x=z}.$$

Here the action of the nabla on b is defined as formal notation:

$$\nabla_\lambda b^{\mu_1\dots\mu_k} = \left\{ {}^{\mu_1}_{\rho\lambda} \right\} b^{\rho\dots\mu_k} + \dots + \left\{ {}^{\mu_k}_{\rho\lambda} \right\} b^{\mu_1\dots\rho}, \qquad (k \in \mathbb{N}_0),$$

i.e. as if b were a tensor independent of x (although it is not a tensor). Then we employ the chain rule for differentiation, so we can write symbolically (with indices omitted):

$$\begin{array}{rcl} I_0(b_0 f) & = & b_0 f(z), \\ I_1(b_1 f) & = & -\left[(\nabla b_1) f + b_1 \nabla f \right]_{x=z}, \\ I_2(b_2 f) & = & \left[(\nabla^2 b_2) f + 2(\nabla b_2)(\nabla f) + b_2 \nabla^2 f \right]_{x=z}, \\ & \vdots & \end{array}$$

Finally, introducing these expressions back into (1.6) and grouping the terms with the same derivative of f we obtain:

$$V[f] = \int_{\mathcal{M}} d^{p+1}\xi \sqrt{-\gamma} \Big[B^{\mu\nu}(\xi) f_{\mu\nu}(z) + B^{\mu\nu\rho}(\xi) \left(\nabla_\rho f_{\mu\nu} \right)_{x=z} + B^{\mu\nu\rho\sigma}(\xi) \left(\nabla_\sigma \nabla_\rho f_{\mu\nu} \right)_{x=z} + \dots \Big], \tag{1.7}$$

where the B coefficients have the following structure:

$$\begin{array}{rcccccccccl} B_0 & = & b_0 & - & \nabla b_1 & + & \nabla^2 b_2 & - & \nabla^3 b_3 & + & \dots, \\ B_1 & = & & - & b_1 & + & 2\nabla b_2 & - & 3\nabla^2 b_3 & + & \dots, \\ B_2 & = & & & & & b_2 & - & 3\nabla b_3 & + & \dots, \\ \vdots & & & & & & & & & & \end{array} \tag{1.8}$$

The diagonal structure of this system of equations allows to be solved for b coefficients:

$$\begin{array}{rcccccccccl} b_0 & = & B_0 & - & \nabla B_1 & + & \nabla^2 B_2 & - & \nabla^3 B_3 & + & \dots, \\ b_1 & = & & - & B_1 & + & 2\nabla B_2 & - & 3\nabla^2 B_3 & + & \dots, \\ b_2 & = & & & & & B_2 & - & 3\nabla B_3 & + & \dots, \\ \vdots & & & & & & & & & & \end{array} \tag{1.9}$$

[1]This assumption does not influence the generality of the result.

Here one defines the action of the nabla on B coefficients in the same way as on b's:

$$\nabla_\lambda B^{\mu_1\dots\mu_k} = \left\{ {\mu_1 \atop \rho\lambda} \right\} B^{\rho\dots\mu_k} + \dots + \left\{ {\mu_k \atop \rho\lambda} \right\} B^{\mu_1\dots\rho}.$$

At this point we have all elements necessary to discuss covariance. First, from (1.7) it follows that the coefficients B transform as proper tensors on $\mathcal{M}$, i.e. according to the rule

$$B'^{\mu_1\dots\mu_k}(\xi) = \left[\frac{\partial x'^{\mu_1}}{\partial x^{\nu_1}} \cdots \frac{\partial x'^{\mu_k}}{\partial x^{\nu_k}} \right]_{x=z} B^{\nu_1\dots\nu_k}(\xi). \tag{1.10}$$

This follows from the quotient law, since $V[f]$ is a scalar with respect to spacetime diffeomorphisms, as is the volume element of the hypersurface $\mathcal{M}$, while $f_{\mu\nu}$ is a tensor. The transformation law for b coefficients can now be read from (1.9), and does not have tensorial character. Also, the action of the nabla on B's now becomes a natural action of a covariant derivative to a tensor that does not depend on x.

Once this is settled, we can rewrite the δ series expansion (1.3) of the field $V^{\mu\nu}(x)$ in manifestly covariant form

$$V^{\mu\nu}(x) = \int_{\mathcal{M}} d^{p+1}\xi \, \sqrt{-\gamma} \left[B^{\mu\nu}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} - \nabla_\rho \left(B^{\mu\nu\rho}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \right) + \dots \right], \tag{1.11}$$

which is most convenient for further use.

1.4 Multipole approximations

As every other expansion into some kind of series. the idea of expansion into δ series becomes really useful only when employed for approximating, by truncating the series at some point. In this case, the concept of truncation is based on neglecting the coefficients standing next to some derivative of the δ function and higher, with an assumption that these coefficients are much smaller than the previous B's. Using the formalism (1.4) this means that one neglects all terms of order $\mathcal{O}_n$ and higher. This allows for the variable $V^{\mu\nu}(x)$ to be modeled in a suitable way to describe matter more or less localized on the surface $\mathcal{M}$.

So for example, neglecting all the terms of order $\mathcal{O}_1$ and higher defines the so-called *single-pole* approximation, and the variable $V^{\mu\nu}(x)$ is written as:

$$V^{\mu\nu}(x) = \int_{\mathcal{M}} d^{p+1}\xi \, \sqrt{-\gamma} \, B^{\mu\nu}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}}.$$

Clearly, this does not mean that everywhere in all equations the derivatives of the δ function should be dropped, but rather that *for the given variable $V^{\mu\nu}(x)$ the coefficients standing next to derivatives* are to be neglected. If this variable is under a derivative in some equation, it must not be neglected, no matter the presence of the derivative of the δ function.

Neglecting all the terms of order $\mathcal{O}_2$ and higher defines the so-called *pole-dipole* approximation, and the variable $V^{\mu\nu}(x)$ is written as:

$$V^{\mu\nu}(x) = \int_{\mathcal{M}} d^{p+1}\xi \, \sqrt{-\gamma} \left[B^{\mu\nu}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} - \nabla_\rho \left(B^{\mu\nu\rho}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \right) \right]. \tag{1.12}$$

Usually it is assumed here that $B^{\mu\nu} \sim \mathcal{O}_0$ and $B^{\mu\nu\rho} \sim \mathcal{O}_1$. Similarly one defines the quadrupole, octupole and higher approximations.

Let us comment also that the truncation is a covariant operation, in the sense that truncation of the series in one coordinate system implies the truncation *at the same point* in all other coordinate systems. This is a consequence of manifestly covariant notation (1.11), and is important for multipole approximations to be well defined.

1.5 Symmetries of multipole approximations

In section 1.3 we studied in detail how the δ series behaves with respect to spacetime diffeomorphisms. We have determined that the notation (1.11) is manifestly covariant with respect to arbitrary coordinate transformation $x \to x' = x'(x)$, and that each variable transforms as a tensor in accordance to spacetime indices it carries. Given that, the B coefficients are special in the sense that after the coordinate transformation the B' coefficient is always evaluated on the surface $\mathcal{M}$.

Similarly one can study the behavior under the reparametrization of the surface $\mathcal{M}$, i.e. diffeomorphisms of ξ coordinates. That analysis is far more simple, and one can easily see that all B coefficients transform as scalars with respect to transformations $\xi \to \xi' = \xi'(\xi)$, since the volume element $d^{p+1}\xi$ together with the factor $\sqrt{-\gamma}$ transforms as a scalar. In complete analogy we can discuss the reparametrization diffeomorphisms on the boundary $\partial\mathcal{M}$, $\lambda \to \lambda' = \lambda'(\lambda)$, and arrive at analogous conclusions.

According to all this, we may conclude that every variable transforms in accordance to its index structure, honoring all three types of indices. This is illustrated in the table.

Transformation	$B^{\mu\nu\rho}(\xi)$	$\gamma_{ab}(\xi)$	$g_{\mu\nu}(x)$	$u^\mu_a(\xi)$	$n_a(\lambda)$	$v^a_i(\lambda)$	$v^\mu_i(\lambda)$
$x \to x' = x'(x)$	tensor	scalar	tensor	vector	scalar	scalar	vector
$\xi \to \xi' = \xi'(\xi)$	scalar	tensor	scalar	vector	vector	vector	scalar
$\lambda \to \lambda' = \lambda'(\lambda)$	scalar	scalar	scalar	scalar	scalar	vector	vector

In order to ensure this manifest covariance also for the derivatives of these variables in various directions, we introduce here also (total) covariant derivatives ∇_a and ∇_i, which act on all variables defined on the surface $\mathcal{M}$ and its boundary $\partial\mathcal{M}$ respectively, correcting every index with an appropriate Christoffel connection term:

$$\begin{aligned} \nabla_a V^{\mu b} &\equiv \partial_a V^{\mu b} + \left\{ {}^{\ \mu}_{\lambda\rho} \right\} u^\rho_a V^{\lambda b} + \left\{ {}^{\ b}_{ca} \right\} V^{\mu c}, \\ \nabla_i V^{\mu bj} &\equiv \partial_i V^{\mu bj} + \left\{ {}^{\ \mu}_{\lambda\rho} \right\} v^\rho_i V^{\lambda bj} + \left\{ {}^{\ b}_{ca} \right\} v^a_i V^{\mu cj} + \left\{ {}^{\ j}_{ki} \right\} V^{\mu bk}. \end{aligned}$$

Christoffel connections on the surface and its boundary are induced as projections of the spacetime connection, and each of them can be calculated as a Christoffel symbol constructed out of appropriate metric, γ_{ab} and h_{ij}. Defined in this way, the derivatives satisfy all metricity conditions:

$$\nabla_a g_{\mu\nu}(z) = 0, \qquad \nabla_a \gamma_{bc}(\xi) = 0,$$

$$\nabla_i g_{\mu\nu}(z(\zeta)) = 0, \qquad \nabla_i \gamma_{ab}(\zeta) = 0, \qquad \nabla_i h_{jk}(\lambda) = 0.$$

After the analysis of spacetime, surface and boundary diffeomorphisms, let us consider two more symmetries, the so-called *first* and *second extra symmetry*. Namely, it turns out that the δ series in addition to diffeomorphisms exhibits some other symmetries, and we shall now study their nature. For simplicity, we shall limit to the pole-dipole approximation of the δ series, although both symmetries can be defined in general.

We may infer the first extra symmetry if we notice that every term in the expansion (1.12) essentially contains $D - p - 1$ δ functions, which localize matter to the surface of a p-brane in D-dimensional spacetime. The remaining $p + 1$ δ functions and corresponding $p + 1$ integrations are there only to make the expression covariant. This suggests that some components of B coefficients are superfluous. Specifically, the derivatives in the directions lying in the surface $\mathcal{M}$ are integrated out, as they should bearing in mind that matter is not localized in these directions. This implies that parallel components of the coefficients $B^{\mu\nu\rho}$ should not appear at all in the expansion (1.12). In order to verify this, define the transformation law of the form

$$\delta_1 B^{\mu\nu\rho} = \epsilon^{\mu\nu a} u^\rho_a, \tag{1.13a}$$

where $\epsilon^{\mu\nu a}(\xi)$ are arbitrary parameters. Demanding that the functional (1.7) remains invariant, it follows that

$$\delta_1 B^{\mu\nu} = \nabla_a \epsilon^{\mu\nu a}, \qquad n_a \epsilon^{\mu\nu a}\Big|_{\partial\mathcal{M}} = 0. \tag{1.13b}$$

Equations (1.13) define the *first extra symmetry*.

It is now easy to see that parallel components of $B^{\mu\nu\rho}$ coefficients are actually pure gauge,

$$\delta_1(B^{\mu\nu\rho}u^a_\rho) = \epsilon^{\mu\nu a},$$

and that they can be gauged away everywhere except on the boundary, where parameters $\epsilon^{\mu\nu a}$ are not completely arbitrary. A consequence of this is that the theory will exhibit some degrees of freedom which live exclusively on the boundary of a p-brane, and do not enter the equations of motion, but rather only boundary conditions. In the following chapter we shall deal with their physical interpretation, among other things.

The second extra symmetry can be inferred if we remember that, exactly speaking, variable $V^{\mu\nu}(x)$ itself does not depend at all on the choice of the surface $\mathcal{M}$ used for expansion into δ series. In other words, if we do not truncate the series, it must remain invariant under the transformations $\mathcal{M} \to \mathcal{M}'$, where $\mathcal{M}'$ denotes some other surface. Naturally, the B coefficients will change accordingly, and their transformation laws under the change $z^\mu \to z'^\mu$ define the gauge symmetry that is named *second extra symmetry*.

Second extra symmetry is exact only if we consider the whole δ series, while truncation breaks it. Indeed, if we truncate the δ series at the n-th term, the condition $B_n = B_{n+1} = \cdots = 0$ becomes a gauge fixing condition which partially fixes the shape of the surface $\mathcal{M}$. If we transform to another surface $\mathcal{M}'$, we must be careful for the new coefficients B'_n, B'_{n+1}, ..., to remain negligible, so that the series is still truncated at the same point. This reduces the possibilities for the choice of the new surface.

All this can be neatly written using the formalism (1.4). The transformation of the surface is defined as

$$z^\mu(\xi) \quad \to \quad z'^\mu(\xi) = z^\mu(\xi) + \epsilon^\mu(\xi), \tag{1.14a}$$

where the parameters ϵ^μ are constrained in pole-dipole approximation by the gauge condition $B'_{n+1} \sim \mathcal{O}_2$ for all $n \in \mathbb{N}$. Requesting the functional (1.7) to remain invariant with respect to transformations (1.14a) gives the laws of transformation for B coefficients,

$$\delta_2 B^{\mu\nu} = -B^{\mu\nu}u^a_\rho\nabla_a\epsilon^\rho - B^{\lambda\nu}\left\{{}^{\;\mu}_{\lambda\rho}\right\}\epsilon^\rho - B^{\mu\lambda}\left\{{}^{\;\nu}_{\lambda\rho}\right\}\epsilon^\rho, \qquad \delta_2 B^{\mu\nu\rho} = -B^{\mu\nu}\epsilon^\rho, \tag{1.14b}$$

and at the same time we see that the gauge condition enforces a restriction on the parameters,

$$\epsilon^\mu(\xi) \sim \mathcal{O}_1. \tag{1.15}$$

Equations (1.14) with the condition (1.15) define the *second extra symmetry*, in pole-dipole approximation.

Regarding these transformations it is necessary to comment three issues. First, note that the ϵ^μ, defined via equation (1.14a), transforms as a vector. Then explicit presence of Christoffel connection in (1.14b) appears to contradict the tensorial character of $B^{\mu\nu}$ coefficients. Of course, this is only apparent, but there is no real contradiction. This is easy to see if we look closely to the transformation law for $B^{\mu\nu}$ coefficients with respect to spacetime diffeomorphisms, (1.10). There the coefficients of the transformation are evaluated at the point $x = z$. When the surface $\mathcal{M}$ changes according to (1.14a), the law of coordinate transformations for the new coefficients $B^{\mu\nu}$ given in (1.14b) is precisely the same as (1.10), where now the coefficients of the transformation are evaluated at the new point, $x = z'$, since the surface has changed. This is possible precisely due to the presence of Christoffel symbols in (1.14b).

The second comment is related to the single-pole approximation. Namely, there gauge conditions $B_1 = B_2 = \cdots = 0$ fix $\epsilon^\mu = 0$ completely, so the symmetry is trivial. This is a consequence of the fact that in this approximation matter is distributed strictly on one surface, so there is no freedom in the choice of $\mathcal{M}$.

Finally, the third comment is related to the fact that fixing the gauge of second extra symmetry defines the so-called *central surface of mass*, which in case of the 0-brane, i.e. the particle, reduces to the familiar notion of center of mass line. In the next chapter we shall see how a good choice of central mass surface simplifies equations of motion of a p-brane and helps us interpret the free parameters of the theory.

With this we end the exposition of general properties of δ series expansion, and we now turn to derivation and analysis of effective equations of motion for a p-brane, using the multipole formalism developed in this chapter.

Chapter 2

EQUATIONS OF MOTION FOR A p-BRANE

As we announced in the foreword, the main subject of this paper is the derivation of effective equations of motion for a p-brane in spacetimes with curvature and torsion. This is done starting from two basic assumptions — Poincaré invariance and the demand for the matter to be localized on the $(p+1)$-dimensional hypersurface $\mathcal{M}$ which is called the brane worldsheet. Poincaré invariance as a consequence gives the covariant conservation laws for the two corresponding currents — the stress–energy tensor and the spin tensor. In the first section we shall deal with the form and properties of these laws. The matter localization assumption enables us to make use of multipole formalism developed in the previous chapter, and we shall mainly work in pole-dipole approximation. For the sake of clarity of the exposition, the derivation of equations of motion itself will first be performed for the special case of scalar matter, in section two. The obtained equations will turn out to be quite complicated and will demand further analysis and interpretation of free parameters, which is taken up in the third section. Finally, in section four we deal with the general case — motion of matter with spin in Riemann-Cartan spacetime, where we upgrade the interpretation from the scalar case with new concepts specific to spin and torsion. The section and the chapter is finalized with the discussion of the single-pole approximation as a special case, because it plays a major role in later analysis of the Dirac matter.

2.1 Covariant conservation laws

One of the main postulates of modern physics is the Poincaré symmetry of physical laws. Specifically, if we split the total Lagrangian $\mathcal{L}$ of a given fundamental theory into the gravitational part $\mathcal{L}_g$ and the rest $\mathcal{L}_m$ (which is by definition called the matter Lagrangian), in virtually all standard theories it turns out that $\mathcal{L}_m$ is itself invariant to local Poincaré transformations. Employing the Emmy Noether theorem, we may then derive covariant conservation laws for the translations current, the so-called "stress–energy tensor" $\tau^{\mu\nu}$, and the Lorentz transformations current, the "spin tensor" $\sigma^{\lambda\mu\nu}$. In our notation, the covariant conservation laws for these two currents have the form [27]:

$$\left(D_\rho + T^\lambda{}_{\rho\lambda}\right)\tau^\rho{}_\mu = \tau^\nu{}_\rho T^\rho{}_{\mu\nu} + \frac{1}{2}\sigma^\nu{}_{\lambda\rho}\mathcal{R}^{\lambda\rho}{}_{\mu\nu}\,, \tag{2.1a}$$

$$\left(D_\rho + T^\lambda{}_{\rho\lambda}\right)\sigma^\rho{}_{\mu\nu} = \tau_{\mu\nu} - \tau_{\nu\mu}\,. \tag{2.1b}$$

These equations are the main starting point in derivation of effective equations of motion for a p-brane. But before we get involved in that task, we need to comment on one very important issue regarding the structure of these equations. At a first glance, the variables in the theory are $\tau^{\mu\nu}$ and $\sigma^{\lambda\mu\nu}$. However, a closer look at the equation (2.1b) reveals that it can be interpreted as an equation for the antisymmetric part of stress–energy tensor, $\tau^{[\mu\nu]}$. Moreover, this equation is *algebraic* in $\tau^{[\mu\nu]}$, and even solved explicitly for it. This means that, if we know $\sigma^{\lambda\mu\nu}$, the antisymmetric part of the stress–energy tensor can be completely calculated, with no arbitrary constants of integration or allowance for boundary conditions typical for differential equations. In other words, $\tau^{[\mu\nu]}$ components do not represent independent variables, but are rather fully determined by $\sigma^{\lambda\mu\nu}$. Because of this we do not treat them as independent, and use (2.1b) to eliminate them entirely from (2.1a), after which we are left with *just one differential equation* for the independent variables $\tau^{(\mu\nu)}$ and $\sigma^{\lambda\mu\nu}$,

$$\nabla_\nu \left(\tau^{(\mu\nu)} + \frac{1}{2} K_{\lambda\rho}{}^{\mu} \sigma^{\nu\lambda\rho} - K^{[\mu}{}_{\lambda\rho} \sigma^{\rho\lambda\nu]} - \nabla_\rho \sigma^{(\mu\nu)\rho} \right) = \frac{1}{2} \sigma_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu}, \tag{2.2}$$

while (2.1b) is considered as a definition of the antisymmetric part of the stress–energy tensor.

Such setup is appropriate for the analysis of kink motion in "external" gravitational field. This means that curvature and torsion are not localized around the p-brane, while *all other fields are*. Of course, one may in principle consider a situation where for example electromagnetic field is also considered to be "external". Then the Lagrangian would be split into the gravitational part $\mathcal{L}_g$, electromagnetic part $\mathcal{L}_{em}$ and the rest $\mathcal{L}_m$ which would be called matter, by definition. In that case, the matter Lagrangian would possess, other than Poincaré symmetry, also the internal gauge symmetry of electromagnetic field, $U(1)$. As a consequence, there would be also another current and a corresponding covariant conservation equation. Of course, it should be noted that the splitting of the Lagrangian into appropriate parts is not unique, due to the interactions between matter and external fields, as well as the interactions between external fields themselves. However, we shall not deal with concrete Lagrangian models at all, but will just suppose that the split has been made somehow, and that the matter fields are localized around a p-brane, while external fields are not. In relation to the program including internal gauge fields, we also need to note that there is a problem in realization of nontrivial coupling of the gauge field with torsion. Namely, it turns out that any such coupling is inconsistent, because it breaks the gauge symmetry. For all these reasons we shall not get into analysis of such cases in this paper.

2.2 Derivation of equations of motion

Let us deal now with the derivation of equations of motion for a p-brane, using the multipole formalism developed in previous chapter. In order to demonstrate the derivation procedure in a clear way, in this and the following section we shall restrict to the case of scalar matter. This means that

$$\sigma^{\lambda\mu\nu} = 0,$$

so the equation (2.2) reduces to

$$\nabla_\nu \tau^{(\mu\nu)} = 0. \tag{2.3}$$

We shall work in pole-dipole approximation, so we expand the stress–energy tensor into a δ series around a $(p+1)$-dimensional worldsheet $\mathcal{M}$:

$$\tau^{(\mu\nu)}(x) = \int_{\mathcal{M}} d^{p+1}\xi \sqrt{-\gamma} \left[B^{\mu\nu}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} - \nabla_\rho \left(B^{\mu\nu\rho}(\xi) \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \right) \right]. \tag{2.4}$$

The derivation of equations of motion goes as follows. First we introduce an arbitrary vector field $f_\mu(x)$ with compact support, and rewrite the equation (2.3) in a suitable form

$$\int d^D x \sqrt{-g} f_\mu \nabla_\nu \tau^{(\mu\nu)} = 0, \qquad \forall f_\mu. \tag{2.5}$$

Then we employ the expansion (2.4) of the stress–energy tensor. Owing to the compactness of the support of the test-function f_μ, we are allowed to commute the integrals and eliminate boundary terms. That leads us to

$$\int d^{p+1}\xi \sqrt{-\gamma} \left(B^{\mu\nu} f_{\mu;\nu} + B^{\mu\nu\rho} f_{\mu;\nu\rho} \right) = 0, \tag{2.6}$$

where $f_{\mu;\nu} \equiv (\nabla_\nu f_\mu)_{x=z}$ i $f_{\mu;\nu\rho} \equiv (\nabla_\rho \nabla_\nu f_\mu)_{x=z}$. The fact that this equation holds for every $f_\mu(x)$ enforces certain restrictions on the coefficients $B^{\mu\nu}$ and $B^{\mu\nu\rho}$. In order to deduce them, it is necessary to rewrite the equation in such a form where only the truly arbitrary and independent derivatives of the field f_μ are present. However, this is nontrivial. Namely, the field $f_\mu(x)$ is indeed arbitrary, but if given everywhere on the surface $\mathcal{M}$, then also all derivatives in the directions tangent to the surface are automatically fixed. Therefore, we split the first and second derivative of f_μ into components orthogonal and parallel to the surface $\mathcal{M}$:

$$f_{\mu;\lambda} = f^{\perp}_{\mu\lambda} + u^a_\lambda \nabla_a f_\mu, \tag{2.7a}$$

$$f_{\mu;(\lambda\rho)} = f^{\perp}_{\mu\lambda\rho} + 2f^{\perp}_{\mu(\lambda a} u^a_{\rho)} + f_{\mu ab} u^a_\lambda u^b_\rho, \tag{2.7b}$$

$$f_{\mu;[\lambda\rho]} = \frac{1}{2} R^\sigma{}_{\mu\lambda\rho} f_\sigma. \tag{2.7c}$$

We obtain these components by the virtue of the projectors

$$P_{\perp}{}^{\mu}_{\nu} = \delta^\mu_\nu - u^\mu_a u^a_\nu, \qquad P_{\parallel}{}^{\mu}_{\nu} = u^\mu_a u^a_\nu. \tag{2.8}$$

Specifically, we have:

$$\begin{array}{lcllcl} f^{\perp}_{\mu\lambda} & = & P_{\perp}{}^{\sigma}_{\lambda} f_{\mu;\sigma}, & f^{\perp}_{\mu\lambda\rho} & = & P_{\perp}{}^{\sigma}_{\lambda} P_{\perp}{}^{\nu}_{\rho} f_{\mu;(\sigma\nu)}, \\ f^{\perp}_{\mu\lambda a} & = & P_{\perp}{}^{\sigma}_{\lambda} u^\nu_a f_{\mu;(\sigma\nu)}, & f_{\mu ab} & = & u^\sigma_a u^\nu_b f_{\mu;(\sigma\nu)}. \end{array}$$

A straightforward calculation now leads us to

$$\begin{array}{lcl} f_{\mu ab} & = & \nabla_{(a}\nabla_{b)} f_\mu - (\nabla_a u^\nu_b) f^{\perp}_{\mu\nu}, \\ f^{\perp}_{\mu\rho a} & = & P_{\perp}{}^{\nu}_{\rho} \nabla_a f^{\perp}_{\mu\nu} + (\nabla_a u^b_\rho)\nabla_b f_\mu + \dfrac{1}{2} P_{\perp}{}^{\lambda}_{\rho} u^\nu_a R^\sigma{}_{\mu\nu\lambda} f_\sigma, \end{array} \tag{2.9}$$

and tells us that the only independent components on the surface are f_μ, $f^{\perp}_{\mu\nu}$ and $f^{\perp}_{\mu\nu\rho}$. At this point we substitute (2.7) and (2.9) into equation (2.6) and group together the coefficients proportional to the independent components of f_μ. The resulting equation has the following general structure:

$$\int_{\mathcal{M}} d^{p+1}\xi \sqrt{-\gamma} \Big[X^{\mu\nu\rho} f^{\perp}_{\mu\nu\rho} + X^{\mu\nu} f^{\perp}_{\mu\nu} + X^\mu f_\mu + \nabla_a \left(X^{\mu\nu a} f^{\perp}_{\mu\nu} + X^{\mu ab} \nabla_b f_\mu + X^{\mu a} f_\mu \right) \Big] = 0.$$

Now owing the fact that f_μ, $f^{\perp}_{\mu\nu}$ and $f^{\perp}_{\mu\nu\rho}$ are independent on the worldsheet, we conclude that the first three terms must separately be equal to zero. This gives us the following equations:

$$P_{\perp}{}^{(\nu}_{\lambda} P_{\perp}{}^{\sigma)}_{\rho} B^{\mu\lambda\rho} = 0, \tag{2.10a}$$

$$P_{\perp}{}^{\sigma}_{\nu} \Big[B^{\mu\nu} - \nabla_a \left(B^{\mu\rho\nu} u^a_\rho + P_{\perp}{}^{\nu}_{\lambda} B^{\mu\lambda\rho} u^a_\rho \right) \Big] = 0, \tag{2.10b}$$

$$\nabla_b \left(B^{\mu\nu} u^b_\nu + 2B^{\mu(\lambda\rho)} u^a_\lambda \nabla_a u^b_\rho - \nabla_a B^{\mu(\lambda\rho)} u^a_\lambda u^b_\rho \right) - \left(P_{\perp}{}^{\rho}_{\sigma} B^{\nu(\lambda\sigma)} + \frac{1}{2} B^{\nu\lambda\rho} \right) R^\mu{}_{\nu\lambda\rho} = 0. \tag{2.10c}$$

What remains is a surface integral which is also zero:

$$\int_{\partial\mathcal{M}} d^p\lambda \sqrt{-h}\, n_a \left(X^{\mu\nu a} f^{\perp}_{\mu\nu} + X^{\mu ab} \nabla_b f_\mu + X^{\mu a} f_\mu \right) = 0. \tag{2.11}$$

When evaluated on the boundary $\partial\mathcal{M}$, components $f^{\perp}_{\mu\nu}$ and f_μ are mutually independent, but $\nabla_a f_\mu$ is not. For this we split the derivative ∇_a into components orthogonal and parallel to the boundary:

$$\nabla_a f_\mu = n_a \nabla_{\perp} f_\mu + v^i_a \nabla_i f_\mu. \tag{2.12}$$

Here we introduced $\nabla_{\perp} \equiv n^a \nabla_a$. Now, $f^{\perp}_{\mu\nu}$, $\nabla_{\perp} f_\mu$ and f_μ are also mutually independent, so the equation (2.11) gives three boundary conditions:

$$P_{\perp}{}^{\nu}_{\lambda} B^{\mu(\lambda\rho)} u^a_\rho n_a \Big|_{\partial\mathcal{M}} = 0, \tag{2.13a}$$

$$B^{\mu\lambda\rho}u^a_\lambda u^b_\rho n_a n_b\Big|_{\partial\mathcal{M}} = 0, \tag{2.13b}$$

$$\Big[\nabla_i\left(B^{\mu(\lambda\rho)}u^a_\lambda u^b_\rho v^i_a n_b\right) - n_b\Big(B^{\mu\nu}u^b_\nu + 2B^{\mu(\lambda\rho)}u^a_\lambda\nabla_a u^b_\rho - \nabla_a B^{\mu(\lambda\rho)}u^a_\lambda u^b_\rho\Big)\Big]\Big|_{\partial\mathcal{M}} = 0. \tag{2.13c}$$

Equations (2.10) and (2.13) describe matter of a p-brane in pole-dipole approximation. As we can see, the basic variables z^μ, $B^{\mu\nu}$ and $B^{\mu\nu\rho}$ enter the equations in such a way that it is quite difficult to give them some physical interpretation. For this reason, we turn to diagonalization of the equations, i.e. rewriting them using new variables which are determined by solving all algebraic equations in (2.10). Namely, it is essential to notice that equation (2.10a) is algebraic, so we use it to switch from $B^{\mu\nu\rho}$ to new variables which satisfy it automatically. splitting $B^{\mu\nu\rho}$ into orthogonal and parallel components and substituting it into (2.10a), we obtain

$$B^{\mu\nu\rho} = 2u^{(\mu}_b B^{\nu)\rho b}_\perp + u^\mu_a u^\nu_b B^{\rho ab}_\perp + u^\rho_a B^{\mu\nu a}, \tag{2.14}$$

while $B^{(\mu\nu)a}_\perp \equiv B^{\mu[ab]}_\perp \equiv B^{[\mu\nu]a} \equiv 0$. Note that the component $B^{\mu\nu a}$ is not split into orthogonal and parallel parts. It is rather left as is because of the first extra symmetry, which suggests that this component is pure gauge and should vanish from the equations of motion.

Now we employ (2.14) to rewrite the next equation of motion, (2.10b), as

$$P_{\perp}{}^{\rho}_{\nu}\Big[B^{\mu\nu} - \nabla_a\left(L^{\mu\nu a} + N^{\mu\nu a}\right)\Big] = 0, \tag{2.15}$$

where

$$L^{\mu\nu a} \equiv B^{\mu\nu a}_\perp + u^{[\mu}_b B^{\nu]ba}_\perp, \qquad N^{\mu\nu a} \equiv B^{\mu\nu a} + u^{(\mu}_b B^{\nu)ba}_\perp. \tag{2.16}$$

The new variables introduced, $L^{\mu\nu a}$ and $N^{\mu\nu a}$ have both orthogonal and parallel components, but will turn out to be very useful as they are. Note here that the definition equation for $L^{\mu\nu a}$ in (2.16) implies the relation

$$L^{\mu\nu[a}u^{b]}_\nu = 0. \tag{2.17}$$

The coefficients $N^{\mu\nu a} = N^{\nu\mu a}$ and $L^{\mu\nu a} = -L^{\nu\mu a}$ are, with condition (2.17), in one-to-one correspondence with $B^{\mu\nu\rho}$:

$$B^{\mu\nu\rho} = 2u^{(\mu}_a L^{\nu)\rho a} + N^{\mu\nu a}u^\rho_a. \tag{2.18}$$

Given this, in what follows we shall eliminate $B^{\mu\nu\rho}$ in favor of $L^{\mu\nu a}$ and $N^{\mu\nu a}$ in all remaining equations.

Now we turn to the analysis of $B^{\mu\nu}$ coefficients. Using the projectors (2.8), we have

$$B^{\mu\nu} = B^{\mu\nu}_\perp + 2u^{(\mu}_b B^{\nu)b}_\perp + u^\mu_a u^\nu_b B^{ab}. \tag{2.19a}$$

Substituting this into (2.15), we obtain

$$B^{\mu\nu}_\perp = P_{\perp}{}^{\mu}_{\lambda}P_{\perp}{}^{\nu}_{\rho}\nabla_a N^{\lambda\rho a}, \qquad B^{\mu a}_\perp = u^a_\lambda P_{\perp}{}^{\mu}_{\rho}\nabla_b\left(L^{\lambda\rho b} + N^{\lambda\rho b}\right), \tag{2.19b}$$

and

$$P_{\perp}{}^{\mu}_{\lambda}P_{\perp}{}^{\nu}_{\rho}\nabla_a L^{\lambda\rho a} = 0. \tag{2.20a}$$

Equations (2.19b) and (2.20a) are equivalent to (2.15). The first tells us that $B^{\mu\nu}_\perp$ and $B^{\mu a}_\perp$ are completely determined via L and N. This means that B^{ab}, $L^{\mu\nu a}$ and $N^{\mu\nu a}$ are the only free parameters in the theory. Also, the equation (2.20a) is called the *angular momentum precession equation*, which will become clear once we give appropriate physical interpretation to coefficients $L^{\mu\nu a}$.

Finally, employ (2.18) and (2.19) to rewrite the only remaining big equation of motion (2.10c) via the new, independent variables:

$$\nabla_b\left(m^{ab}u^\mu_a - 2u^b_\lambda\nabla_a L^{\mu\lambda a} + u^\mu_c u^c_\rho u^b_\lambda\nabla_a L^{\rho\lambda a}\right) - u^\nu_a L^{\lambda\rho a}R^\mu{}_{\nu\lambda\rho} = 0, \tag{2.20b}$$

where

$$m^{ab} \equiv B^{ab} - u^a_\rho u^b_\lambda\nabla_c N^{\rho\lambda c}. \tag{2.21}$$

Equation (2.20b) represents a differential equation to determine the functions $z^\mu(\xi)$, so it is called the *equation of motion of a p-brane*. The variable m^{ab} is a symmetric tensor on the worldsheet, and we use it instead of B^{ab} as a free parameter. As we anticipated using the first extra symmetry, the coefficients $N^{\mu\nu a}$ dropped out of equations (2.20).

Now, using all these results, we perform an analogous diagonalization procedure on the boundary conditions (2.13). Using algebraic relations (2.18), (2.19) and (2.21), we rewrite the boundary conditions via the independent coefficients:

$$L^{\mu\nu a} n_a n_\nu \Big|_{\partial\mathcal{M}} = 0, \tag{2.22a}$$

$$P_{\perp}{}^{\mu}_{\lambda} P_{\perp}{}^{\nu}_{\rho} L^{\lambda\rho a} n_a \Big|_{\partial\mathcal{M}} = 0, \tag{2.22b}$$

$$\Big[\nabla_i \left(N^{ij} v^\mu_j + 2L^{\mu\nu a} n_a v^i_\nu \right) - n_b \left(m^{ba} u^\mu_a - 2u^b_\nu \nabla_a L^{\mu\nu a} + u^\mu_c u^c_\rho u^b_\lambda \nabla_a L^{\rho\lambda a} \right) \Big] \Big|_{\partial\mathcal{M}} = 0, \tag{2.22c}$$

where

$$N^{ij} \equiv N^{\mu\nu a} n_a v^i_\mu v^j_\nu. \tag{2.23}$$

The coefficients N^{ij} are defined only on the boundary $\partial\mathcal{M}$, and do not appear anywhere else in the equations. This was also anticipated in the discussion of the first extra symmetry, because on the boundary there was a restriction for the gauge parameters, which reduced the number of degrees of freedom available to be gauged away on the boundary.

Let us sum up the results The dynamics of a p-brane in pole-dipole approximation is determined via:

- the equation of motion (2.20b),

$$\nabla_b \left(m^{ab} u^\mu_a - 2u^b_\lambda \nabla_a L^{\mu\lambda a} + u^\mu_c u^c_\rho u^b_\lambda \nabla_a L^{\rho\lambda a} \right) - u^\nu_a L^{\lambda\rho a} R^\mu{}_{\nu\lambda\rho} = 0,$$

- the angular momentum precession equation (2.20a),

$$P_{\perp}{}^{\mu}_{\lambda} P_{\perp}{}^{\nu}_{\rho} \nabla_a L^{\lambda\rho a} = 0,$$

- the boundary conditions (2.22) which those two equations must satisfy.

These equations are equivalent to the covariant conservation equation (2.3).

It is necessary to make a few comments. First, in the equations appear the free parameters $L^{\mu\nu a}$, m^{ab} and N^{ij}, which carry the information about the internal structure of the brane, i.e. about the properties of the matter the brane is made of. In the next section we shall analyze these coefficients and give them the physical interpretation of the angular momentum current and the effective stress–energy tensors of the brane and its boundary, respectively.

Second, we see that torsion is entirely absent from the equations. This is a consequence of the assumption that matter carries no spin, and may also be seen from the starting equation (2.2) where torsion couples exclusively with spin.

The third comment is in regard to interactions. Namely, we see that there is an explicit interaction of parameters $L^{\mu\nu a}$ with both the curvature tensor and the orbit of the brane. In the particle special case the interaction with the curvature is responsible for the geodesic deviation, i.e. the violation of the weak equivalence principle. Of course, this is not a surprise since the particle in pole-dipole approximation is not "completely pointlike", but exhibits some thickness characterized by the parameters $L^{\mu\nu a}$, so "tidal effects" appear, i.e. geodesic deviation. In the case of the string and other branes this effect is also present. As for the interaction with the orbit, in the particle special case it can be eliminated using the second extra symmetry, as we shall see in the next chapter. This is related to the choice of the center-of-mass line, and is characteristic for the particle case. In the case of a string and other branes the second extra symmetry may also be utilized to fix the appropriate central surface of mass, but that is not enough to eliminate the interaction with the orbit, which remains present in the equations.

Fourth, we should comment on the transformation laws for the free parameters. According to definitions (2.16), (2.21) and (2.23) we see the following:

- under the diffeomorphisms of spacetime, the worldsheet $\mathcal{M}$ and its boundary $\partial\mathcal{M}$, parameters m^{ab}, $L^{\mu\nu a}$ and $N^{\mu\nu a}$ (so also N^{ij}) transform as tensors, in accordance with their index structure.
- under the first extra symmetry, according to definition (1.13) we have:

$$\delta_1 m^{ab} = 0, \qquad \delta_1 L^{\mu\nu a} = 0, \qquad \delta_1 N^{\mu\nu a} = \epsilon^{\mu\nu a},$$

where on the boundary the parameters $\epsilon^{\mu\nu a}$ satisfy the restriction (1.13b), so $\delta_1 N^{ij} = 0$. In other words, the equations of motion (2.20) and boundary conditions (2.22) are invariant under the first extra symmetry, as expected. Also, we see that coefficients $N^{\mu\nu a}$ are pure gauge everywhere but on the boundary, where components N^{ij} survive.

- under the second extra symmetry, using (1.14) and splitting the parameters $\epsilon^{\mu} = \epsilon^{\mu}_{\perp} + u^{\mu}_{a}\epsilon^{a}$, we obtain:

$$\delta_2 m^{ab} = -\left(u^{c}_{\mu} m^{ab} + u^{(a}_{\mu} m^{b)c}\right) \nabla_c \epsilon^{\mu}_{\perp} + \left(\epsilon^c \nabla_c m^{ab} - 2m^{(bc}\nabla_c \epsilon^{a)}\right),$$

$$\delta_2 L^{\mu\nu a} = -m^{ab} u^{[\mu}_{b} \epsilon^{\nu]}_{\perp}, \qquad \delta_2 N^{ij} = -m^{ab} v^{i}_{a} v^{j}_{b} n_c \epsilon^c. \tag{2.24}$$

Of course, equations of motion and boundary conditions are invariant under this symmetry as well, taking into account the transformation laws of geometric variables.

In regard to the second extra symmetry it should be noted that parameters ϵ^a represent the moving of the surface into itself, given that tangential motions actually do not change the shape of the surface. This holds everywhere but on the boundary, and it can be seen that the subgroup of the second extra symmetry, defined as

$$z^{\mu} \to z'^{\mu} = z^{\mu} + u^{\mu}_{a}\epsilon^{a}, \qquad \epsilon^a n_a \Big|_{\partial\mathcal{M}} = 0,$$

is actually identical to surface reparametrizations, $\xi^a \to \xi'^a = \xi^a + \epsilon^a(\xi)$, since $\epsilon^a \sim \mathcal{O}_1$, so the quadratic and higher terms may be neglected in pole-dipole approximation. Because of this we could in principle eliminate all terms containing ϵ^a from the above equations. Nevertheless, on the boundary the parameters ϵ^a still need not obey the restriction $\epsilon^a n_a = 0$, so in general we must leave them present in the equations.

Now it is a proper moment to understand the physical meaning of the parameters m^{ab}, $L^{\mu\nu a}$ and N^{ij}, in order to make an easier path to the understanding the general case of matter with spin.

2.3 Analysis and physical interpretation

Let us now deal with the interpretation of the coefficients m^{ab}, N^{ij} and $L^{\mu\nu a}$. First, the m^{ab} coefficients appear already in single-pole approximation [24, 25], and represent an *effective* $(p+1)$*-dimensional stress–energy tensor* of a p-brane. Let us repeat briefly known results — in single-pole approximation the mass tensor is covariantly conserved, $\nabla_a m^{ab} = 0$, and in the case of a particle coincides with its total mass. In the string case we may perform an eigenvalue problem analysis in order to classify four basic inequivalent types of matter the string is made of: massive, massless, Nambu-Goto and tachyonic type. In general case of a p-brane we can also use similar methods to classify matter, and find even more inequivalent types of matter. All these results remain the same also in pole-dipole approximation, with the exception of covariant conservation which is broken by higher order terms.

The appearance of strange N^{ij} coefficients which live exclusively on the boundary is a consequence of the boundary condition in (1.13b) which the first extra symmetry parameters must obey. Physically, the N^{ij} coefficients characterize the tangential component of the p-brane thickness. Namely, when an infinitely thin brane is thickened, which essentially happens when going from single-pole to pole-dipole approximation, the thickening is performed in all spatial directions. Obviously, thickening in the directions tangential to the brane surface changes nothing in its interior, simply because the brane is not localized in those directions anyway. However, if the brane has a spatial boundary, thickening at those points is a nontrivial change. The boundary structure obtained in this way is characterized with N^{ij} coefficients.

Actually, they represent a p-dimensional stress–energy tensor of the brane boundary (of the order $\mathcal{O}_1$), in full analogy with m^{ab} for the whole brane. The best way to see this is to examine the case of a p-brane with an extra $(p-1)$-brane attached to its boundary (this sort of situation will be studied for the $p = 1$ example in section 3.2). The procedure for derivation of equations of motion then gives modified boundary conditions where N^{ij} coefficients appear as an $\mathcal{O}_1$ correction terms to the effective stress–energy tensor of the boundary brane, m^{ij} (which is of the order $\mathcal{O}_0$). If that extra boundary-brane is absent, N^{ij} coefficients become dominant, and reflect the fact that the boundary has "thickness". Thus we see that thickening of the brane has an effect amounting to attaching a "light" $(p-1)$-brane to the boundary.

Turning now to coefficients $L^{\mu\nu a}$, we note that there are several ways to see that these may be interpreted as the brane angular momentum current. The first one is the generalization from the case of 0-brane, where these coefficients are already interpreted as particle angular momentum [4]. The second one is straightforward computation of the angular momentum tensor $M^{\mu\nu\rho} \equiv \tau^{\rho[\mu}x^{\nu]}$. However, in this paper we shall employ the third method, which amounts to simple counting of the charges for the $(p+1)$-currents $L^{\mu\nu a}$, and then demonstrating that there are precisely as many of them as there are generators of the appropriate rotation group.

First fix an appropriate coordinate system. Pick an arbitrary point on the brane, and equip it with inertial coordinate systems of spacetime and worldsheet. In this way, $g_{\mu\nu}$ and γ_{ab} reduce at that point to $\eta_{\mu\nu}$ and η_{ab}, respectively. Then, by a suitable Lorentz rotation of the spacetime basis we fix the reference frame to be comoving for that point:

$$u^\mu_a = \delta^\mu_a \,.$$

In this gauge, the relation (2.17) reduces to

$$L^{\mu ab} = L^{\mu ba} \,. \tag{2.25a}$$

Now we count the number of independent charge densities $L^{\mu\nu 0}$. As a first step, we make use of the relation (2.25a), and the antisymmetry

$$L^{\mu\nu a} = -L^{\nu\mu a} \tag{2.25b}$$

to eliminate coefficients L^{abc}. We are left with charge densities $L^{\bar\mu\bar\nu 0}$ and $L^{\bar\mu a 0}$ (here we use a "bar" notation to split the spacetime indices as $\mu = (a, \bar\mu)$). Since $\bar\mu$ takes $D-p-1$ values, there are $(D-p-1)(D-p-2)/2$ independent $L^{\bar\mu\bar\nu 0}$ coefficients and $(D-p-1)(p+1)$ independent $L^{\bar\mu a 0}$ coefficients. In total, this makes

$$\frac{D(D-1)}{2} - \frac{(p+1)p}{2} \equiv \dim\left[SO(1, D-1)\right] - \dim\left[SO(1,p)\right] \tag{2.26a}$$

independent charge densities $L^{\mu\nu 0}$.

As we can see, the number of independent charges of the currents $L^{\mu\nu a}$ is given as a difference of two terms. The first represents the dimension of the $SO(1, D-1)$ group, or equivalently, the number of independent Lorentz rotations in D spacetime dimensions. The second term is the dimension of the $SO(1,p)$ group, i.e. the number of independent Lorentz rotations on the $(p+1)$-dimensional worldsheet. So, our charges correspond to Lorentz rotations orthogonal to the worldsheet. Given that, we naturally associate them with the internal angular momentum of the brane.

It should be noticed that among the charges $L^{\mu\nu 0}$ there is not a single one corresponding to tangential, i.e. rotations on the worldsheet. The reason for this lies in the fact that such rotations are already accounted for through the effective stress–energy tensor of the brane, m^{ab}. Indeed, these rotations do not require nontrivial brane thickness — they are present even in the single-pole approximation. Contrary to these, the possibility of matter rotating in the orthogonal planes in a comoving frame demands brane thickness, i.e. at least pole-dipole approximation. The coefficients $L^{\mu\nu a}$ give some measure of this thickness, and do not exist in the case of infinitely thin brane. As a consequence, the $p(p+1)/2$ components of angular momentum related to tangential rotations are rather associated to the m^{ab} currents than to $L^{\mu\nu a}$.

In order to consider this even further, let us introduce an additional splitting of the worldsheet indices, $a = (0, \bar a)$, which separates time components. Now the nonzero densities of $L^{\mu\nu a}$ coefficients are written as $L^{\bar{\mu}\bar{\nu}0}$, $L^{\bar{\mu}\bar{a}0}$ and $L^{\bar{\mu}00}$. They correspond to $\bar{\mu}-\bar{\nu}$, $\bar{\mu}-\bar{a}$ and $\bar{\mu}-0$ planes of rotation, respectively. Specifically, $L^{\bar{\mu}\bar{\nu}0}$ and $L^{\bar{\mu}\bar{a}0}$ correspond to spatial rotations, while $L^{\bar{\mu}00}$ correspond to boosts.

At this point we may employ the freedom of second extra symmetry to fix some unphysical degrees of freedom. In order to do this, first apply the transformation laws (2.24) to boosts $L^{\bar{\mu}00}$ in the comoving reference frame $u^\mu_a = \delta^\mu_a$. The resulting equation

$$\delta_2 L^{\bar{\mu}00} = \frac{1}{2} m^{00} \epsilon^{\bar{\mu}}$$

says that the boosts $L^{\bar{\mu}00}$ are pure gauge and can be eliminated. This leaves us with the densities of the spatial angular momentum densities $L^{\bar{\mu}\bar{\nu}0}$ and $L^{\bar{\mu}\bar{a}0}$ as only physical charge densities for the currents $L^{\mu\nu a}$. As above, via direct counting we see that there are precisely

$$\frac{(D-1)(D-2)}{2} - \frac{p(p-1)}{2} \equiv \dim\left[SO(D-1)\right] - \dim\left[SO(p)\right] \tag{2.26b}$$

independent charges. They correspond to *spatial rotations orthogonal to the brane*.

According to the equations of motion (2.20), we see that the currents $L^{\mu\nu a}$ couple not only to spacetime curvature, but also with the orbit of the p-brane. In the case of a free particle this interaction can be completely gauged away using the second extra symmetry, i.e. via suitable choice of the center-of-mass line. It turns out that in flat spacetime this line is precisely a straight line, and this shall be demonstrated when we examine the particle example in detail.

After the appropriate interpretation is given to the free parameters m^{ab}, N^{ij} and $L^{\mu\nu a}$, we turn now to the general case of motion of matter with nonzero spin.

2.4 Equations of motion for matter with spin

In section 2.1 we have established that, for the description of matter with spin, it is most convenient to use the variables $\tau^{(\mu\nu)}$ and $\sigma^{\lambda\mu\nu}$, which satisfy the covariant conservation law (2.2). In the previous two sections we have studied the special case of spinless matter in a torsionless spacetime, in the pole-dipole approximation. Now we turn to the discussion of the general case.

Begin with the definition of the approximation we shall adopt:

$$\begin{aligned} \tau^{(\mu\nu)}(x) &= \int_{\mathcal{M}} d^{p+1}\xi\sqrt{-\gamma}\left[B^{\mu\nu}(\xi)\frac{\delta^{(D)}(x-z)}{\sqrt{-g}} - \nabla_\rho\left(B^{\mu\nu\rho}(\xi)\frac{\delta^{(D)}(x-z)}{\sqrt{-g}}\right)\right], \\ \sigma^{\lambda\mu\nu}(x) &= \int_{\mathcal{M}} d^{p+1}\xi\sqrt{-\gamma}\, C^{\lambda\mu\nu}(\xi)\frac{\delta^{(D)}(x-z)}{\sqrt{-g}}. \end{aligned}$$

Let us focus on several important comments. As a first comment, observe that we work in the pole-dipole approximation for $\tau^{(\mu\nu)}$, while simultaneously in the single-pole approximation for the spin tensor. In what follows we shall call this choice simply the pole-dipole approximation, despite the fact that it is not fully such. There are four reasons for introducing this kind of approximation. First, one can note that in the conservation law (2.2) the spin tensor among other places appears under an additional derivative, in the final term on the left. From there one can see that the symmetrized $C^{(\mu\nu)\rho}$ will precisely add up with the $B^{\mu\nu\rho}$ coefficients, thus giving the spin contribution to the angular momentum current $L^{\mu\nu a}$, as we shall see below. In this way one naturally introduces the *total angular momentum* current, which will be convenient for the subsequent analysis, and it is also visible that the monopole term of the spin tensor naturally combines with the dipole term of the stress–energy tensor. Second, this approximation provides a good enough framework for the discussion of the *proper* single-pole approximation, which is obtained by the choice $B^{\mu\nu\rho} \equiv 0$, and which is much more important and interesting in its consequences. Third, an eventual contribution of the dipole term in the spin tensor which we could include would drastically complicate the equations and does not have an interesting physical interpretation. Finally, as we shall see in the examples, there are interesting indications that the monopole term of the spin tensor is of the order $\mathcal{O}_1$, i.e. of the same order as the dipole term of the stress–energy tensor. For all these reasons, we will not discuss the dipole spin term, a practice that follows all previous papers in this area.

The second important comment is the behavior of the two extra symmetries in this approximation. The first extra symmetry is a consequence of the fact that we have nonphysical degrees of freedom in the dipole term of the stress–energy tensor, and it exists here in the same form. Let us just note that the monopole spin term transforms trivially with respect to this symmetry, as expected.

The second extra symmetry is, by its nature, connected to the approximation order, which in our case makes the situation slightly more complicated. As we have already seen, the stress–energy tensor in the pole-dipole approximation has this symmetry provided that $B^{\mu\nu} \sim \mathcal{O}_0$ and $B^{\mu\nu\rho} \sim \epsilon^\mu \sim \mathcal{O}_1$. However, there is an issue with the spin tensor, since it is specified in the single-pole approximation. Namely, if its monopole coefficient $C^{\lambda\mu\nu}$ is considered as a quantity of order $\mathcal{O}_0$, the symmetry is lost, since the invariance of the spin tensor then requires $\epsilon^\mu \equiv 0$, as expected for the single-pole approximation. If we want to conserve the second extra symmetry, we can correct this with the requirement that $C^{\lambda\mu\nu} \sim \mathcal{O}_1$. However, the price we have to pay is the impossibility of studying the matter whose spin is of the same order as its energy, loosely speaking. This is easy to see because $B^{\mu\nu} \sim \mathcal{O}_0 \gg \mathcal{O}_1 \sim C^{\lambda\mu\nu}$. This property is something we always need to keep in mind in the situations when it is important to fix the second extra symmetry.

After the discussion of all the issues related to the setup, we turn to the derivation of the equations of motion. Given that the equation (2.2) is drastically more complicated than (2.3), the equations are somewhat more cumbersome than those of section 2.2, so we will not present all the details. The method of derivation is, however, completely analogous. One multiplies the equation (2.2) with a compact support test function $f_\mu(x)$, and integrate over spacetime in order to eliminate all δ functions. Then all derivatives of f_μ are split into orthogonal and tangential components in order to extract independent components $f^\perp_{\mu\nu\rho}$, $f^\perp_{\mu\nu}$ and f_μ. After all remaining partial integrations, one can write the result in the form:

$$\int_{\mathcal{M}} d^{p+1}\xi\sqrt{-\gamma}\Big[X^{\mu\nu\rho}f^\perp_{\mu\nu\rho} + X^{\mu\nu}f^\perp_{\mu\nu} + X^\mu f_\mu + \nabla_a\left(X^{\mu\nu a}f^\perp_{\mu\nu} + X^{\mu ab}\nabla_b f_\mu + X^{\mu a}f_\mu\right)\Big] = 0.$$

Since the components $f^\perp_{\mu\nu\rho}$, $f^\perp_{\mu\nu}$ and f_μ are mutually independent and arbitrary, the corresponding coefficients X are equated to zero and one thus obtains the equations of motion and boundary conditions analogous to (2.10) and (2.13).

The next step is the diagonalization of these equations. As (2.10a) before, the key role is played by the equation $X^{\mu\nu\rho} = 0$, which now looks as:

$$P_\perp{}^{(\nu}_{\lambda} P_\perp{}^{\sigma)}_{\rho}\Big[B^{\mu\lambda\rho} + C^{(\mu\lambda)\rho}\Big] = 0. \tag{2.27}$$

Given that it has formally the same form as (2.10a), the corresponding general solution can be written as:

$$B^{\mu\nu\rho} + C^{(\mu\nu)\rho} = 2u^{(\mu}_a J^{\nu)\rho a} + N^{\mu\nu a}u^\rho_a, \tag{2.28}$$

where now $J^{\mu\nu a}$ and $N^{\mu\nu a}$ satisfy the relations

$$J^{\mu\nu[a}u^{b]}_\nu = 0, \qquad J^{\mu\nu a} = -J^{\nu\mu a}, \qquad N^{\mu\nu a} = N^{\nu\mu a}.$$

The quantity $N^{\mu\nu a}$ is again a pure gauge with respect to the first extra symmetry, while the quantity $J^{\mu\nu a}$ will become the *total angular momentum* once we discuss the form of the remaining equations.

Further diagonalization of the equations $X^{\mu\nu} = 0$, $X^\mu = 0$ and boundary conditions is performed in a way analogous to the section 2.2 and we will skip the details, but it is important to note the following — in all the equations $B^{\mu\nu\rho}$ never appears on its own, but *always* in the combination $B^{\mu\nu\rho} + C^{(\mu\nu)\rho}$. This allows us to use (2.28) to eliminate that sum in favor of variables J and N. Of course, the coefficient $C^{\mu\nu\rho}$ does appear on its own in other places, so we treat it as an independent variable.

Finally, once the diagonalization procedure is completed, we obtain the following results:

- the equation of motion for the p-brane:

$$\nabla_b\Big[m^{ab}u^\mu_a - 2u^b_\lambda\left(\nabla_a J^{\mu\lambda a} + D^{\mu\lambda}\right) + u^\mu_c u^c_\rho u^b_\lambda\left(\nabla_a J^{\rho\lambda a} + D^{\rho\lambda}\right)\Big] = \tag{2.29}$$

$$= u^\nu_a J^{\lambda\rho a} R^\mu{}_{\nu\lambda\rho} + \frac{1}{2} C_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu},$$

- the angular momentum precession equation:

$$P_\perp{}^\mu_\lambda P_\perp{}^\nu_\rho \left(\nabla_a J^{\lambda\rho a} + D^{\lambda\rho}\right) = 0, \tag{2.30}$$

- the boundary conditions:

$$J^{\mu\nu a} n_\nu n_a\Big|_{\partial\mathcal{M}} = 0, \qquad P_\perp{}^\mu_\lambda P_\perp{}^\nu_\rho n_a J^{\lambda\rho a}\Big|_{\partial\mathcal{M}} = 0,$$

$$\nabla_i\left(N^{ij}v^\mu_j + 2J^{\mu\nu a}n_a v^i_\nu\right)\Big|_{\partial\mathcal{M}} = \tag{2.31}$$

$$= n_b\Big[m^{ab}u^\mu_a - 2u^b_\rho\left(\nabla_a J^{\mu\rho a} + D^{\mu\rho}\right) + u^\mu_c u^c_\sigma u^b_\nu\left(\nabla_a J^{\sigma\nu a} + D^{\sigma\nu}\right)\Big]\Big|_{\partial\mathcal{M}}.$$

Here we have introduced a shorthand

$$D^{\mu\nu} \stackrel{\text{def}}{\equiv} K^{[\mu}{}_{\lambda\rho}C^{\rho\lambda\nu]} + \frac{1}{2}K_{\lambda\rho}{}^{[\mu}C^{\nu]\rho\lambda}.$$

The equations above represent *equations of motion for a p-brane made of matter with spin in the gravitational field with curvature and torsion, in the pole-dipole approximation*, and represent the main result of this paper.

Free parameters used to describe the matter are m^{ab}, N^{ij}, $J^{\mu\nu a}$ and $C^{\lambda\mu\nu}$. The parameters m and N have the same interpretation of effective stress–energy tensor of the p-brane and its boundary, like in the case of scalar matter in Riman spacetime. Coefficients C simply describe the spin tensor current, so it remains only to provide an appropriate interpretation for J. This is done as follows — if we choose $C^{\lambda\mu\nu} = 0$, the equations reduce to (2.10) and (2.13), provided that $J^{\mu\nu a}$ is identified with the orbital angular momentum current $L^{\mu\nu a}$. However, if we choose torsion to be zero, $K^{\mu\nu\rho} = 0$, and leave nonzero spin tensor, we see that the equations again reduce to (2.10) and (2.13), provided that now C does not appear on its own anywhere, but only through J. Given that J couples to the curvature and to the orbit of the p-brane in the same way as the orbital angular momentum current, but now has a contribution of the spin current which precisely adds up with the orbit current as per (2.28), the most natural interpretation for $J^{\mu\nu a}$ is the *total angular momentum current*. This current has properties identical to those from the previous section. Moreover, if we assume that $C^{\lambda\mu\nu} \sim \mathcal{O}_1$, the second extra symmetry allows one to gauge away the boosts of the total angular momentum in the same way as was previously done for the orbital part. This completes the interpretation of the equations of motion and the free parameters of the p-brane in the pole-dipole approximation.

The single-pole case. Let us now turn to the important special case, the single-pole approximation. It is defined by eliminating the dipole term from the stress–energy tensor, $B^{\mu\nu\rho} \equiv 0$. However, this has far reaching consequences for the permitted form of the spin tensor. Namely, B vanishes from the equation (2.27). Solving this equation as before, with the condition of antisymmetry $C^{\lambda\mu\nu} = -C^{\lambda\nu\mu}$ we obtain the following general form for the spin tensor:

$$C^{\lambda\mu\nu} = 2u^\lambda_a S^{\mu\nu a} - E^{\lambda ab}u^\mu_a u^\nu_b + S^{\lambda\mu\nu}. \tag{2.32}$$

Here $S^{\mu\nu a}$, $E^{\mu ab}$ and $S^{\lambda\mu\nu}$ (of course with m^{ab}) are the only remaining free parameters of the p-brane. By definition, they satisfy the following conditions:

$$S^{\mu\nu[a}u^{b]}_\nu = 0, \qquad S^{\mu\nu a} = -S^{\nu\mu a}, \qquad E^{\mu ab} = -E^{\mu ba}, \qquad S^{\lambda\mu\nu} = -S^{\lambda\nu\mu} = S^{\nu\lambda\mu}.$$

Comparing (2.28) and (2.32) we find the connection between the old and new coefficients,

$$J^{\mu\nu a} = S^{\mu\nu a}, \qquad N^{\mu\nu a} = E^{(\mu ab} u_b^{\nu)},$$

so the equations of motion and the boundary conditions for the single-pole approximation are obtained from (2.29), (2.30) and (2.31) by a simple substitution

$$J^{\mu\nu a} \to S^{\mu\nu a}, \qquad N^{ij} \to E^{ij} \equiv E^{\mu ab} n_a v_\mu^{(i} v_b^{j)}.$$

This means that the equations of motion and boundary conditions in the single-pole approximation have basically the same form as in the pole-dipole approximation, with the important additional condition (2.32). Because of this we will not write them explicitly. The interpretation of the parameters is also not necessary, since m^{ab} is still the effective stress–energy tensor of the p-brane, while $S^{\mu\nu a}$, $S^{\lambda\mu\nu}$ and $E^{\mu ab}$ are simply various component currents of the spin tensor (for which there are no common names).

What must be emphasized regarding the single-pole approximation is its meaning — elimination of every "perpendicular extended" structure, in the sense of the thickness of the p-brane. Among other things, this means also the elimination of all orthogonal components of the orbital angular momentum $L^{\mu\nu a}$, which will be especially interesting in the case of the particle, since in that case all components of the orbital angular momentum are orthogonal and it is thus completely eliminated.

Finally, let us comment on one important property of the equation (2.32) in the case of the particle. First, given that in the particle case all Latin indices take only one value, zero, the E coefficients are identically zero due to the antisymmetry of appropriate indices. Thus in the particle case the equation (2.32) becomes

$$C^{\lambda\mu\nu} = 2u^\lambda S^{\mu\nu} + S^{\lambda\mu\nu}, \tag{2.33}$$

where $S^{\mu\nu}$ and $S^{\lambda\mu\nu}$ are totally antisymmetric tensors. This equation is different from the corresponding equation found in the literature [10, 11, 12]. Namely, all authors who have discussed the particle in this regime have obtained an equation of this type, but without the coefficient $S^{\lambda\mu\nu}$. Since the spin tensor of the Dirac particle is totally antisymmetric, such must be the monopole coefficients $C^{\lambda\mu\nu}$. However, imposing the antisymmetry to the equation $C^{\lambda\mu\nu} = 2u^\lambda S^{\mu\nu}$ (which can be found throughout the previous literature) it is easy to show that $S^{\mu\nu} = 0$, and thus $C^{\lambda\mu\nu} = 0$, and finally as a consequence $\sigma^{\lambda\mu\nu} = 0$ as well. In other words, the total antisymmetry of the spin tensor appeared to be forbidden, which was interpreted by the authors in [10, 11, 12] that the *Dirac particle cannot be discussed in the single-pole approximation*.

If we study in more detail the root of the difference between the equation obtained by those authors and our equation (2.33), it turns out that it is due to the fact that those authors have treated the antisymmetric part of the stress–energy tensor $\tau^{[\mu\nu]}$ as an *independent variable*. The consequence of this was that in the single-pole approximation the equation (2.1b) enforces a strong constraint on $\sigma^{\lambda\mu\nu}$, which effectively eliminates precisely the totally antisymmetric component $S^{\lambda\mu\nu}$ from (2.33). But, if one takes into account the analysis of the first section of this chapter, from the discussion of the independent variables of the equations (2.1) one can see that such a restriction of the coefficients $C^{\lambda\mu\nu}$ is completely unjustified. Moreover, in this way one recovers also the case of the Dirac particle, since (2.33) allows that the total antisymmetry of the spin tensor. Because of this, and the fact that precisely Dirac particle is the major candidate for the measurement of the torsion, in the next chapter we will study in detail the case of the Dirac c particle in the single-pole approximation, and discover a whole series of very interesting results.

Chapter 3

EXAMPLES

Let us now turn to the analysis of the special cases and examples which demonstrate the variety of the dynamics following from our equations of motion. We will begin with the particle, and then turn to the string, concentrating mostly on the realistic case of 4-dimensional spacetime.

3.1 Particle

The particle represents the case of a 0-brane, and it sweeps out a worldline in spacetime. We parametrize this line with a single parameter ξ^0, and it can be chosen such that in every point of the worldline the induced metric is $\gamma_{00} \equiv \gamma = -1$. This is a common gauge condition, and the coordinate ξ^0 is usually denoted τ and called *proper time*. Since Latin indices $a, b, c, \dots$ in this case take only one value, zero, we can eliminate them from the equation of motion and the angular momentum precession equation. Those two equations can then be written as:

$$\nabla\Big[mu^\mu + 2u_\lambda \left(\nabla J^{\mu\lambda} + D^{\mu\lambda}\right)\Big] = u^\nu J^{\lambda\rho} R^\mu{}_{\nu\lambda\rho} + \frac{1}{2} C_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu},$$

and

$$P_\perp{}^\mu_\lambda P_\perp{}^\nu_\rho \left(\nabla J^{\lambda\rho} + D^{\lambda\rho}\right) = 0.$$

There are no boundary conditions, since the worldline has no boundary. The scalar m is called *mass*, while $J^{\mu\nu}$ is the total angular momentum of the particle.

Particle in the pole-dipole approximation. In order to illustrate the meaning of the second extra symmetry, let us consider the motion of the particle with the assumption that $C^{\lambda\mu\nu} \sim J^{\mu\nu} \sim \mathcal{O}_1$. In analogy to (2.24) we calculate the variation of the boost component of the angular momentum, $J^{\mu\nu}u_\nu$, with respect to the second extra symmetry:

$$\delta_2 \left(J^{\mu\nu} u_\nu\right) = \frac{m}{2} \epsilon^\mu_\perp.$$

From here one can see that this component can be gauged away, and it follows that $J^{\mu\nu} = J^{\mu\nu}_\perp$. Once we have established this, contracting the equation of motion with u_μ we obtain

$$\nabla m = D^\mu_\perp \nabla u_\mu - \frac{1}{2} C_{\nu\rho\lambda} \nabla K^{\rho\lambda\nu} \sim \mathcal{O}_1,$$

where we have used that $D^{\mu\nu} \equiv D^{\mu\nu}_\perp + D^{[\mu}_\perp u^{\nu]}$. Substituting this back into the equation of motion we then obtain that $\nabla u^\mu \sim \mathcal{O}_1$, which can be used further since in the pole-dipole approximation we neglect terms of order $\mathcal{O}_2$. After a short calculation, we obtain the equations of motion

$$\nabla\Big(mu^\mu - D^\mu_\perp\Big) = u^\nu J^{\lambda\rho}_\perp R^\mu{}_{\nu\lambda\rho} + \frac{1}{2} C_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu}, \qquad \nabla J^{\mu\nu}_\perp + D^{\mu\nu}_\perp = 0,$$

while the derivative of the mass reduces to

$$\nabla m = -\frac{1}{2} C_{\nu\rho\lambda} \nabla K^{\rho\lambda\nu} \sim \mathcal{O}_1 .$$

A few comments are in order. First, we see that the spin-orbit interaction has completely vanished. What remains are the interactions between the total angular momentum with curvature and spin with torsion. Second, angular momentum and mass are not conserved, precisely because of the spin–torsion interaction.

If there is no torsion in spacetime, these equations reduce to

$$m\nabla u^\mu = u^\nu J_\perp^{\lambda\rho} R^\mu{}_{\nu\lambda\rho}, \qquad \nabla J_\perp^{\mu\nu} = 0, \qquad \nabla m = 0.$$

Mass and angular momentum are conserved, and the equation of motion differs from a geodesic only due to the interaction of the angular momentum with curvature. Since the spin couples exclusively with torsion (except for the piece present in the total angular momentum), the equations of motion for the scalar particle in the spacetime with curvature and torsion will look the same as these, knowing that in that case $J_\perp^{\mu\nu} \equiv L_\perp^{\mu\nu}$.

Finally, if the spacetime features no curvature and no torsion, the equation of motion reduces to a straight line.

Particle in the single-pole approximation. Let us now discuss a much more interesting situation, namely the particle in spacetime with curvature and torsion in the single-pole regime. This regime differs from the above analysis in the assumption that now $C^{\lambda\mu\nu} \sim \mathcal{O}_0$, so we do not have the possibility to fix the gauge of the second extra symmetry. The equations of motion have the form

$$\nabla\Big[mu^\mu + 2u_\lambda \left(\nabla S^{\mu\lambda} + D^{\mu\lambda} \right) \Big] = u^\nu S^{\lambda\rho} R^\mu{}_{\nu\lambda\rho} + \frac{1}{2} C_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu},$$

$$P_{\perp}{}^\mu_\lambda P_{\perp}{}^\nu_\rho \left(\nabla S^{\lambda\rho} + D^{\lambda\rho} \right) = 0,$$

where now we also have the relation (2.33):

$$C^{\lambda\mu\nu} = 2u^\lambda S^{\mu\nu} + S^{\lambda\mu\nu}.$$

From here we see that a *point particle with spin does not follow a geodesic*, both because the spin-orbit interaction and because of the interaction of spin with curvature and torsion. However, we can discuss the important special case of the Dirac particle, which suggests that one should be careful with this conclusion about the trajectory. Namely, it can turn out that the field equations for matter place *additional restrictions on the kink solution*, which would imply that $C^{\lambda\mu\nu} \sim \mathcal{O}_1$. This can be neglected in the single-pole approximation, which means that the equation of motion still reduces to the geodesic equation. In order to see that such a possibility exists, let us study in detail the case of a Dirac particle.

Dirac particle case. One of the basic characteristics of the Dirac field is that its spin tensor $\sigma^{\lambda\mu\nu}$ is totally antisymmetric. This property is inherited by the coefficients $C^{\lambda\mu\nu}$, so from (2.33) one can obtain a key result:

$$S^{\mu\nu} = 0. \tag{3.1}$$

This result has far reaching consequences. Substituting it into the equation of motion and the spin precession equation, we obtain

$$\nabla \left(mu^\mu + K^{[\mu} s^{\nu]} u_\nu \right) + \frac{1}{2} s^\nu \nabla^\mu K_\nu = 0, \qquad K_\perp^{[\mu} s_\perp^{\nu]} = 0,$$

where we have introduced the spin vector s^μ as $S^{\mu\nu\rho} \equiv e^{\mu\nu\rho\lambda} s_\lambda$, as well as the axial component of the contorsion $K^\mu \equiv e^{\mu\nu\rho\lambda} K_{\nu\rho\lambda}$. It is immediately visible from the equation of motion that both spin-orbit and spin-curvature interactions have vanished. Only the interaction of the spin with the axial component of the contorsion has remained. The particle trajectory is still not a geodesic, but only because of that remaining interaction. We can therefore conclude that *in torsionless spacetime the Dirac particle follows a geodesic*. Regarding the spin precession equation, it transforms into an *algebraic* constraint between the orthogonal components of spin and contorsion, which requires that the spin be always oriented in the direction of the external field $K^{\mu}_{\perp}$. This is, to put it mildly, a weird behavior of the Dirac particle, and one obvious (though not the only) way to resolve this situation is based on the assumption that the spin of the Dirac particle in the single-pole approximation should be *neglected*. In order to see if this is really a legitimate assumption, it is necessary to construct a concrete model in the framework of some concrete theory.

As a simplest candidate for the analysis, let us choose the theory of free Dirac field in Minkowski spacetime. The Lagrangian is:

$$\mathcal{L} = \frac{i}{2}\left[\bar{\psi}\gamma^{\mu}\partial_{\mu}\psi - \left(\partial_{\mu}\bar{\psi}\gamma^{\mu}\psi\right)\right] - m\bar{\psi}\psi.$$

Dirac γ matrices are defined so that they satisfy the standard anticommutation relations $\{\gamma^{\mu},\gamma^{\nu}\} = -2\eta^{\mu\nu}$, while $\gamma_5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$. Since this is a free field theory, there are no self-interactions and therefore no kink solutions. Because of this we model the particle as a wave packet, which represents a configuration of the Dirac field well localized in space, while inside it still resembles a plane wave. The size of the packet ℓ is considered in the limit $\ell \to 0$, so that we would implement the idea of the single-pole approximation. The wave packet dissolves as time passes, but in the limit $\lambda/\ell \to 0$ the speed of the dissolution can be considered small (λ represents the dominant wavelength of the packet). For concreteness, let us construct the wave packet as follows. At the initial moment $t = 0$, we choose the following field configuration,

$$\psi(\vec{r},0) = Ae^{-\frac{r^2}{\ell^2}}\psi_p(\vec{r},0)$$

where

$$\psi_p(x) \equiv \sqrt{\frac{k^0+m}{2m}}\begin{bmatrix} 1 \\ 0 \\ \frac{k^3}{k^0+m} \\ 0 \end{bmatrix} e^{ik_{\mu}x^{\mu}}$$

is one solution of the Dirac equation $(i\gamma^{\mu}\partial_{\mu} - m)\,\psi = 0$, i.e. a plane wave which travels along the x^3-axis (here $k^1 = k^2 = 0$ and $k^0 \equiv \sqrt{m^2 + (k^3)^2}$) while its spin is polarized "up", for concreteness. The spin factor is written in the conventional Dirac representation of γ-matrices. The exponential function multiplying the plane wave in $\psi(\vec{r},0)$ serves to cut out one small piece and to define its size ℓ, while A represents the overall amplitude of the packet. The wavelength of the packet, λ, corresponds to the wavelength of the plane wave, $1/|\vec{k}|$.

Using the Dirac equation we can calculate the time derivatives of this field configuration, and thus determine its evolution. However, we are actually interested to calculate the currents $\tau^{(\mu\nu)}$ and $\sigma^{\lambda\mu\nu}$, which depend mostly on the first derivatives:

$$\tau_{\mu\nu} = i\left[\bar{\psi}\gamma_{\mu}\partial_{\nu}\psi - \left(\partial_{\nu}\bar{\psi}\right)\gamma_{\mu}\psi\right] - 2\eta^{\mu\nu}\mathcal{L}, \qquad \sigma^{\lambda\mu\nu} = \varepsilon^{\lambda\mu\nu\rho}\bar{\psi}\gamma_5\gamma_{\rho}\psi. \tag{3.2}$$

Eliminating the time derivatives using the Dirac equation, a straightforward calculation gives the expressions for both currents at the moment $t = 0$,

$$\tau^{(00)} = -2|A|^2 e^{-\frac{2r^2}{\ell^2}}\frac{(k^0)^2}{m}, \qquad \tau^{(33)} = -2|A|^2 e^{-\frac{2r^2}{\ell^2}}\frac{(k^3)^2}{m}, \tag{3.3a}$$

$$\tau^{(0\alpha)} = -2|A|^2 e^{-\frac{2r^2}{\ell^2}}\frac{k^0}{m}\left(k^3\eta^{3\alpha} - \frac{x_{\beta}}{\ell^2}\varepsilon^{\alpha\beta 3}\right), \tag{3.3b}$$

$$\sigma^{123} = -|A|^2 e^{-\frac{2r^2}{\ell^2}}\frac{k^3}{m}, \qquad \sigma^{012} = -|A|^2 e^{-\frac{2r^2}{\ell^2}}\frac{k^0}{m}, \tag{3.3c}$$

while all other components vanish. Integrating these expressions over the 3-dimensional space we can obtain the monopole coefficients $B^{\mu\nu}$ and $C^{\lambda\mu\nu}$, while multiplying with x^α and integrating we obtain corresponding dipole coefficients, and so on. The nonzero monopole coefficients are

$$B^{00} = -|A|^2\sqrt{\frac{\pi^3}{2}}\ell^3\frac{(k^0)^2}{m}, \qquad B^{33} = -|A|^2\sqrt{\frac{\pi^3}{2}}\ell^3\frac{(k^3)^2}{m}, \qquad B^{03} = -|A|^2\sqrt{\frac{\pi^3}{2}}\ell^3\frac{k^0k^3}{m},$$

$$C^{123} = -\frac{1}{2}|A|^2\sqrt{\frac{\pi^3}{2}}\ell^3\frac{k^3}{m}, \qquad C^{012} = -\frac{1}{2}|A|^2\sqrt{\frac{\pi^3}{2}}\ell^3\frac{k^0}{m},$$

while the sole nonzero dipole coefficient is

$$B^{012} = -B^{021} = -\frac{1}{4}|A|^2\sqrt{\frac{\pi^3}{2}}\ell^3\frac{k^0}{m}.$$

Quadrupole and higher terms are proportional to $|A|^2\ell^4k^3$, to $|A|^2\ell^4(k^3)^2$, or to higher degrees of ℓ.

Now let us construct the limit which corresponds to to the single-pole approximation. First, we see that the monopole and dipole terms are of the order ℓ^3 while the quadrupole and higher terms are of higher order. The amplitude A can be chosen such that in the limit $\ell \to 0$ monopole terms are of order of unity,

$$|A|^2 = \frac{const}{\ell^3(k^3)^2} \equiv \frac{const}{\ell^3}\lambda^2,$$

since $k^3 \sim k^0 \sim \lambda^{-1}$. Now we have

$$B^{00} \sim B^{33} \sim B^{03} \sim 1, \qquad C^{123} \sim C^{012} \sim B^{012} \sim \lambda,$$

while quadrupole and higher multipoles are of the order ℓ, $\ell\lambda$ and higher in ℓ. The limit $\ell \to 0$ eliminates all of them and implements the pole-dipole approximation, because only monopole and dipole terms remain. However, we have already noted that the stability of the wave packet can be established only if $\lambda \ll \ell$, since only then the interior of the packet still resembles a plane wave. In other words, the limit $\ell \to 0$ implies also $\lambda \to 0$, so we see that dipole terms also vanish. Thus the limit $\ell \to 0$ establishes the single-pole approximation.

At this point we arrive to the crucial insight — the spin monopole terms have vanished together with the orbital dipole terms in this limit. The reason for this is easy to see, since $2B^{012} = C^{012}$, which represents a *constraint between the monopole spin and the dipole stress-energy moment*. The consequence of this is that the spin does not participate in the effective equation of motion for the wave packet, which eliminates the spin-orbit interaction and we are left with the equation for the straight line.

This example explicitly demonstrates that $C^{\lambda\mu\nu} \sim \mathcal{O}_1$ for the free Dirac particle. This property is completely independent from the geometry of spacetime, so we can conclude that the *point particle with spin* $1/2$ *follows a geodesic in the space with curvature and torsion*. Of course, this conclusion has been demonstrated on an example, but it can be generalized in the following way. One can easily see that the constraint $2B^{012} = C^{012}$ is simply a single-pole approximation of a more general relation, which the currents (3.3) satisfy *exactly*:

$$\tau^{(0\alpha)} = \frac{1}{2}\partial_\beta\sigma^{0\alpha\beta}, \qquad \alpha = 1, 2.$$

Covariantization of this equation leads to a very interesting requirement for the currents:

$$\tau^{\mu\nu}j_\nu \propto j^\mu, \tag{3.4}$$

where $j^\mu \equiv \bar\psi\gamma^\mu\psi$ is the current of the $U(1)$ symmetry for the Dirac field, and in the above example it is proportional to the wave vector k^μ. From this requirement and the total antisymmetry of the Dirac spin tensor $\sigma^{\lambda\mu\nu}$ one can derive in the comoving frame $j^\alpha = 0$ that

$$x^\alpha\tau^{(0\beta)} - x^\beta\tau^{(0\alpha)} = \sigma^{0\alpha\beta} + div.$$

Integrating over 3-dimensional space, the divergence term vanishes while the remainder reduces to the statement that the spin monopole moment is equal to the orbital dipole moment, and therefore of the order $\mathcal{O}_1$. In other words, if a given theory has a kink solution which satisfies the condition (3.4), its spin must be of the order $\mathcal{O}_1$, from which we conclude that the kink in the single-pole approximation moves along a geodesic.

The equation (3.4) can also be used as a criterion for the non-Dirac fields. Since all fields (except the real scalar field) have at least the $U(1)$ symmetry, there exists a corresponding current j^μ, so the condition (3.4) is well defined. In the comoving system we can now derive again the relation between the spin and orbital angular momentum, but since the spin tensor does not need to be totally antisymmetric, the relation has a slightly different form:

$$x^\alpha \tau^{(0\beta)} - x^\beta \tau^{(0\alpha)} = \sigma^{[\beta 0\alpha]} + x^{[\alpha}\partial_0 \sigma^{00\beta]} + div.$$

The integration over a 3-dimensional space removes the divergence, while the second term on the right-hand side becomes the dipole term of the spin tensor, and as such already of the order $\mathcal{O}_1$ or higher, so one can again conclude that the spin monopole moment is of the order $\mathcal{O}_1$, i.e. negligible in the single-pole approximation. From this one can conclude that the spin-orbit, spin-curvature and spin-torsion interaction terms all vanish from the equation of motion, which reduces it again to a geodesic equation. Of course, all this is valid under the assumption that (3.4) actually holds for the given kink solution of a given field, which need not be the case. If this equation is not satisfied, the trajectory can in principle be different from a geodesic.

At the end of this analysis we can speculate that perhaps the condition (3.4) is satisfied *always* when the matter is considered in the single-pole approximation. Apparently, there is nothing particularly special in the statement that all currents of a point particle flow in the direction of its motion, which includes the stress–energy tensor $\tau^{\mu\nu}$. This would mean that the test point particles always follow geodesics, regardless of the spin, curvature and torsion. Of course, such a speculation is an open problem which might deserve further research, but we will not pursue it further in this work.

This concludes the analysis of the equations of motion of a particle. In what follows we turn to the analysis of the motion of a string and other extended objects.

3.2 String

The trajectory of a string, i.e. a 1-brane, is a two-dimensional worldsheet with a one-dimensional boundary (if there is one). Similarly to the particle case, the boundary can be parametrized with the proper time τ, while the boundary indices $i, j, \dots$ take only the zero value, so we will not write them explicitly. The induced metric on the boundary, h, and the corresponding tangent vector v^a satisfy

$$h = v^a v_a = -1\,.$$

The only particularity of the string dynamics, compared to other branes, is the possibility to employ the second extra symmetry in order to gauge away the coefficients N^{ij} which live on the string boundary. Of course, in order for this to be possible, we have to assume that the coefficients of the spin tensor $C^{\lambda\mu\nu}$ are of the order $\mathcal{O}_1$. Since the boundary is one-dimensional, we have only one component $N \equiv N^{00}$, and its variation is

$$\delta_2 N = -m^{ab} v_a v_b \epsilon\,,$$

where $\epsilon \equiv \epsilon^a n_a$ is the only free gauge parameter at the boundary. From this we see that we can fix the gauge $N = 0$, while the parameters ϵ^a satisfy the restriction $\epsilon^a n_a|_{\partial\mathcal{M}} = 0$.

In what follows we will discuss some concrete types and configurations of a string, in order to illustrate dynamics described by the equations of motion.

Nambu-Goto string. Let us first turn to the important special case of the motion of the Nambu-Goto string in spacetime with curvature and torsion. Nambu-Goto string is by definition an infinitely thin spinless string, with the effective stress–energy tensor given as $m^{ab} = T\gamma^{ab}$, where T is a constant called the string tension.

The condition that the string is infinitely thin allows us to work in the single-pole approximation. As we have already explained in the discussion of the single-pole approximation, the equations of motion and boundary conditions are obtained from (2.29), (2.30) and (2.31) substituting $J^{\mu\nu a} \to S^{\mu\nu a}$ and $N^{ij} \to E^{ij} \equiv E^{\mu ab} n_a v_\mu^{(i} v_b^{j)}$. In this way the big equation of motion (2.29) becomes

$$\nabla_b \Big[m^{ab} u^\mu_a - 2u^b_\lambda \left(\nabla_a S^{\mu\lambda a} + D^{\mu\lambda} \right) + u^\mu_c u^c_\rho u^b_\lambda \left(\nabla_a S^{\rho\lambda a} + D^{\rho\lambda} \right) \Big] =$$

$$= u^\nu_a S^{\lambda\rho a} R^\mu{}_{\nu\lambda\rho} + \frac{1}{2} C_{\nu\rho\lambda} \nabla^\mu K^{\rho\lambda\nu}.$$

The condition that the string has no spin reduces the equation further into the form

$$\nabla_b \left(m^{ab} u^\mu_a \right) = 0.$$

Finally, the choice of the mass tensor $m^{ab} = T\gamma^{ab}$ reduces the equation of motion to the familiar equation for the extremal surface

$$\nabla_a u^{\mu a} = 0.$$

We see that the explicit interaction with the curvature of spacetime vanishes, while the only remaining one is the implicit interaction through the definition of the covariant derivative. Also, torsion has completely vanished from the equation, since it couples only with spin. In other words, the *Nambu-Goto string does not feel the presence of torsion, while the presence of curvature manifests itself exclusively through Christoffel connection.*

The spin precession equation

$$P_\perp{}^\mu_\rho P_\perp{}^\nu_\sigma \left(\nabla_a S^{\rho\sigma a} + D^{\rho\sigma} \right) = 0,$$

is homogeneous in the spin tensor, so it is identically satisfied if there is no spin. What remains are the boundary conditions,

$$S^{\mu\nu a} n_\nu n_a \Big|_{\partial\mathcal{M}} = 0, \qquad P_\perp{}^\mu_\lambda P_\perp{}^\nu_\rho n_a S^{\lambda\rho a} \Big|_{\partial\mathcal{M}} = 0,$$

$$\nabla \left(E v^\mu - 2S^{\mu\nu a} n_a v_\nu \right) \Big|_{\partial\mathcal{M}} =$$

$$= n_b \Big[m^{ab} u^\mu_a - 2u^b_\rho \left(\nabla_a S^{\mu\rho a} + D^{\mu\rho} \right) + u^\mu_c u^c_\sigma u^b_\nu \left(\nabla_a S^{\sigma\nu a} + D^{\sigma\nu} \right) \Big] \Big|_{\partial\mathcal{M}},$$

which are also identically satisfied when $C^{\lambda\mu\nu} = 0$, except the last one which reduces to the form

$$\gamma^{ab} n_b u^\mu_a \Big|_{\partial\mathcal{M}} = 0.$$

This is precisely the standard Neumann boundary condition for the free Nambu-Goto string. We can therefore see that our equations contain all information about the Nambu-Goto string as their special case.

Rigid rotating rod. Let us now study a massive rod which rotates around its longitudinal axis. For simplicity, we will work in Minkowski spacetime ($R^\mu{}_{\nu\lambda\rho} = 0$ and $T^\lambda{}_{\mu\nu} = 0$), and in Cartesian coordinates ($g_{\mu\nu}(x) = \eta_{\mu\nu}$).

We seek a simple solution of the equations of motion which describes the rod at rest along the x-axis between the points $x = L/2$ and $x = -L/2$. We want it to rotate around its longitudinal axis, so we choose

$$J^a \equiv J^{23a} = -J^{32a}$$

to be the only nonzero components of the total angular momentum current. We can fix the coordinates of the worldsheet ξ^a using a gauge condition for the reparametrization symmetry $z^a(\xi) = \xi^a$, while the boundary coordinate λ matches the proper time τ. As a consequence of all this, we have:

$$u^\mu_a = \delta^\mu_a\,, \qquad v^a = \delta^a_0\,, \qquad \gamma_{ab} = \eta_{ab}\,, \qquad h = -1\,.$$

It is easy to verify that this represents a solution of the equations of motion (2.29), (2.30) and the boundary conditions (2.31), provided that

$$\partial_a m^{ab} = 0\,, \qquad \partial_a J^a = 0\,, \tag{3.5a}$$

and

$$m^{a1}(\xi^1 = \pm\tfrac{L}{2}) = 0\,, \qquad J^1(\xi^1 = \pm\tfrac{L}{2}) = 0\,. \tag{3.5b}$$

Equations (3.5a) tell us that the effective stress–energy tensor m^{ab} and the angular momentum current J^a are conserved quantities. Equations (3.5b) tell us that there is no flow of energy, momentum and angular momentum through the end-points of the rod.

The only thing that is maybe not completely obvious in this example is that the rod is actually rotating around its longitudinal axis. In order to verify this, let us compute the total angular momentum of the whole rod:

$$M^{\mu\nu} \equiv \frac{1}{2}\int d^3x\, \left(x^\mu \tau^{(0\nu)} - x^\nu \tau^{(0\mu)}\right), \tag{3.6}$$

from where we obtain

$$M^{23} = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx J^0(t,x)\,, \qquad M^{12} = M^{13} = 0\,.$$

Therefore, the rod really rotates in the $y - z$ plane, i.e. around the x-axis. Besides this, the total energy of the rod, defined as

$$E = \int d^3x\; \tau^{00}\,, \tag{3.7}$$

matches its total mass:

$$E = \int_{-\frac{L}{2}}^{\frac{L}{2}} dx\; m^{00}(t,x)\,.$$

The absence of the rotational kinetic energy is the consequence of the fact that $J^a \sim \mathcal{O}_1$. Indeed, the rotational energy is quadratic in J^a, which contributes to the total energy via an $\mathcal{O}_2$ term, which is neglected in the pole-dipole approximation. Let us note at the end of this example that the total angular momentum $M^{\mu\nu}$ and the total energy E are conserved charges. This immediately follows from the equations (3.5a) and the boundary conditions (3.5b).

A simple model of a meson. In this example we will consider a Nambu-Goto string with no internal angular momentum, with two massive particles connected to its ends. This can be understood as a very simple classical model of a meson, in which the quark-antiquark pair is described with the two particles, while the QCD gluon field connecting them is described via a Nambu-Goto string. For simplicity, we will again work in Minkowski spacetime.

The stress–energy tensor is given as a sum of two terms,

$$\tau^{(\mu\nu)} = \tau_{\rm s}^{(\mu\nu)} + \tau_{\rm p}^{(\mu\nu)}, \tag{3.8a}$$

where

$$\tau_{\rm s}^{(\mu\nu)} = \int_{\mathcal{M}} d^2\xi \sqrt{-\gamma} B^{\mu\nu}_{\rm s} \frac{\delta^{(4)}(x-z)}{\sqrt{-g}}, \tag{3.8b}$$

$$\tau_{\rm p}^{(\mu\nu)} = \int_{\partial\mathcal{M}} d\lambda \sqrt{-h}\left[B^{\mu\nu}_{\rm p}\frac{\delta^{(4)}(x-z)}{\sqrt{-g}} - \nabla_\rho\left(B^{\mu\nu\rho}_{\rm p}\frac{\delta^{(4)}(x-z)}{\sqrt{-g}}\right)\right]. \tag{3.8c}$$

The first part of the stress–energy tensor describes the string, without internal angular momentum. Since we want the string to be of Nambu-Goto type, we choose $\sigma_{\rm s}^{\lambda\mu\nu} = 0$ and we turn off the dipole term in $\tau_{\rm s}^{(\mu\nu)}$, while for the mass tensor we choose $m^{ab} = T\gamma^{ab}$. This gives us the standard Nambu-Goto equation of motion

$$\nabla^a u^\mu_a = 0. \tag{3.9}$$

The second part of the stress–energy tensor describes the particle in the pole-dipole approximation, with the restriction that its trajectory matches the boundary of the string. The boundary conditions for the string can then be reinterpreted as the equations of motion for the two particles:

$$p_\perp{}^\mu_\lambda \, p_\perp{}^\nu_\rho \frac{d}{d\tau} J^{\lambda\rho} = 0 \,, \tag{3.10a}$$

$$\frac{d}{d\tau}\left(m v^\mu + 2 v_\nu \frac{d}{d\tau} J^{\mu\nu} \right) = T n^a u^\mu_a. \tag{3.10b}$$

Here we have chosen again the gauge $h = -1$, $\lambda \equiv \tau$ on the boundary, while $p_\perp{}^\mu_\nu \equiv \delta^\mu_\nu + v^\mu v_\nu$ represents the orthogonal projector of the string boundary, and it should not be mixed with $P_\perp{}^\mu_\nu$. The boundary conditions (3.10) are different from the equations of motion for the free particle both by the presence of the spin-orbit interaction (which cannot be gauged away this time) and by the presence of a term on the right-hand side which represents the force which the string exerts on the particle. Let us note that these equations of motion are valid for each of the two particles individually, and that the masses and angular momenta of the two particles are a priori different, since the boundary of the string worldsheet consists by assumption of two nonintersecting curves.

Next, we will impose the following condition to the angular momenta of the particles,

$$J^{\mu\nu} v_\nu = 0, \tag{3.11}$$

which eliminates the boosts. Physically, this condition means that the center of mass of the particle matches the end-point of the string, up to $\mathcal{O}_2$ precision. If we were not to impose this condition, we would allow for the two particles to "swing" on the string. This is physically acceptable, but in this example we are not interested in such type of motion. Given that this condition reduces the number of independent components of the angular momentum to three, we can introduce new notation,

$$\vec{J} \equiv \varepsilon^{0\lambda\rho\mu} J_{\lambda\rho} \vec{e}_\mu$$

which collects these three components into one spacelike 3-vector.

Let us look for a simple solution of the string equation of motion (3.9), where the string has the shape of a *straight line*. With no loss of generality, we can choose coordinates x^μ and ξ^a such that the equation of the world sheet has the form

$$\vec{z} = \vec{\alpha}(t)\sigma, \qquad z^0 = t,$$

where $\xi^0 \equiv t$ and $\xi^1 \equiv \sigma$ have the domains $(-\infty, \infty)$ and $[-1, 1]$, respectively. Assuming also that the length of the string $L \equiv 2|\vec{\alpha}|$ and the velocity of the string end-points $V \equiv |d\vec{\alpha}/dt|$ are constant, the equation (3.9) reduces to

$$\frac{d^2}{dt^2}\vec{\alpha} + \omega^2 \vec{\alpha} = 0, \qquad \omega \equiv \frac{2V}{L} \,.$$

This equation describes uniform rotation in a plane. Choosing the latter to be the $x - y$ plane, we easily obtain the solution

$$\vec{\alpha} = \frac{L}{2} \left(\cos\omega t \, \vec{e}_x + \sin\omega t \, \vec{e}_y \right). \tag{3.12}$$

Consider now the boundary conditions (3.10). Without getting into the details of the straightforward but cumbersome calculation, by solving these equations we obtain that the particle's angular momentum satisfies

$$\frac{d\vec{J}}{dt} = 0, \qquad \vec{J} = J\vec{e}_z, \tag{3.13}$$

while its velocity is

$$V = \frac{1}{\sqrt{1+\frac{2\mu}{TL}}}, \qquad \mu \equiv m + \sqrt{\frac{2T}{mL}} J. \tag{3.14}$$

Since each particle has its own mass and angular momentum, we will denote them as $m_\pm$ and $J_\pm$ for the particle at the point $\sigma = \pm 1$, respectively. Given that both particles have the same speed, their masses must satisfy the condition $\mu_+ = \mu_- \equiv \mu$. We see that the masses themselves, $m_\pm$, are different despite the fact that the center of mass for the whole system is at the string's middle-point $\sigma = 0$. This is a consequence of the nontrivial spin-orbit interaction which contributes to the total energy of the system.

The analysis of the expression (3.14) gives that $V < 1$, as expected. In the limit $\mu \to 0$, the ends of the string move with the speed of light, describing the dynamics of the Nambu-Goto string with the Neumann boundary conditions. In the limit $\mu \to \infty$, the ends of the string are at rest, which represents the example of its dynamics with the Dirichlet boundary conditions.

Total angular momentum and energy of the whole system can be calculated as in the previous example, using the general formulas (3.6) and (3.7). A straightforward calculation gives:

$$E = TL\frac{\arcsin V}{V} + \frac{2\mu}{\sqrt{1-V^2}} - \frac{2V}{L}\left(J_+ + J_-\right),$$

$$M = \frac{TL^2}{4}\left(\frac{\arcsin V}{V^2} - \frac{\sqrt{1-V^2}}{V}\right) + \frac{2\mu}{\sqrt{1-V^2}}\frac{LV}{2} + J_+ + J_-.$$

These results have obvious interpretation. The total energy of the system consists of the energy of the string, energies of the two particles, and the energy of the spin-orbit interactions, respectively. The rotational kinetic energy of the particles, being quadratic in $\vec{J}$, is negligible in the pole-dipole approximation. Angular momentum of the system consists of the piece describing the rotation of the string, the piece which describes the revolution of the two particles, and finally their internal angular momenta.

If we consider the limit in which the masses of the particles are small, the free parameter L can be eliminated from these above two equations in favor of the total energy E, which leads us to the constraint

$$M = \frac{1}{2\pi T}E^2 + 2\left(J_+ + J_-\right). \tag{3.15}$$

The first term on the right-hand side defines the well-known law of Regge trajectories, while the second term represents a small correction (of order $\mathcal{O}_1$) due to the presence of particles with nonzero angular momenta at the ends of the string. As one can see, the unique Regge trajectory from the standard string theory description splits into a whole family of different trajectories. This is an interesting effect, since in experiments we actually observe that different types of particles are grouped on different trajectories.

Finally, let us discuss also the single-pole approximation of this example. We have already assumed this approximation for the string, so its equation of motion (3.9) remains the same. As for the particles, the equations of motion (3.10) now become

$$p_{\perp}{}^{\mu}_{\lambda}\, p_{\perp}{}^{\nu}_{\rho} \frac{d}{d\tau} S^{\lambda\rho} = 0\,,$$

$$\frac{d}{d\tau}\left(mv^\mu + 2v_\nu \frac{d}{d\tau} S^{\mu\nu}\right) = Tn^a u^\mu_a,$$

where the spin tensor of the particle is constrained by a condition of the type (2.33):

$$C^{\lambda\mu\nu} = 2v^\lambda S^{\mu\nu} + S^{\lambda\mu\nu}.$$

Under the assumption that $S^{\mu\nu}v_\nu = 0$ all the results of this example remain the same, with the substitution $J \to S$. However, an interesting possibility is the model of the meson consisting of two quarks.

Quarks are Dirac particles, so their spin tensor is totally antisymmetric, and as a consequence we again have the result (3.1):

$$S^{\mu\nu} = 0.$$

All nonzero components of the spin of the Dirac particle are inside the $S^{\lambda\mu\nu}$ term, which is completely decoupled from the above equations of motion. We have seen this already in the analysis of the Dirac particle in the previous section, where the spin was coupled exclusively with torsion. Repeating the derivation of the Regge trajectories law (3.15) we now obtain

$$M = \frac{1}{2\pi T} E^2,$$

since the component of the spin which would enter the correction term is actually equal to zero. In other words, *in the single-pole approximation there is no splitting of the Regge trajectory into a family, for Dirac particles*. Of course, since we can see more than one such trajectory in experiment, we conclude that the single-pole approximation is insufficient to describe the splitting effect. This can also be understood as an indication (suggested by the example) that the quarks inside the meson cannot be treated as point particles. Of course, this example is just one very gross model, and further analysis in necessary to verify these conclusions more reliably.

3.3 Conclusions

In this paper we have studied the motion of extended objects in an external gravitational field with torsion. The natural formalism which allows for the analysis of these problems and the derivation of the equations of motion is called the multipole formalism and it describes the matter localized around some p-brane in spacetime. The main computational tool used for this is related to the expansion of a given function into a power series of the derivatives of the Dirac δ function around that p-brane. The δ expansion represents a powerful apparatus, mainly because it allows for a manifestly covariant notation, provides the possibility for the systematical analysis of all the symmetries in the theory, and because it works for the arbitrary dimension p of the p-brane. This last property is very important, since it establishes the main original contribution of this research — derivation of the effective equations of motion not only for pointlike, but also for the extended objects.

Because of all this, the first chapter of this work is devoted to the multipole formalism. After a short introduction to the Riemann-Cartan geometry, we have introduced the notion of the expansion of a given function into a power series in the derivatives of the δ function, written it in the manifestly covariant way. Next we have defined the single-pole and pole-dipole approximations which consist of the truncation of the δ expansion beyond the first and second term, respectively. The chapter was concluded with the analysis of the symmetries which exist in the pole-dipole approximation, where besides the spacetime and reparametrization diffeomorphisms we have found two extra gauge symmetries. The first reflects the fact that in the covariant notation we have surplus variables, while the second reflects the freedom for the choice of the surface around which the δ expansion is being performed.

The second chapter concentrated on the application of the multipole formalism to the description of matter in the external gravitational field. We have started from two fundamental assumptions — that the matter Lagrangian is invariant with respect to the local Poincare group, and that the matter is localized around some p-dimensional hypersurface $\mathcal{M}$ in spacetime. The first assumption enforces the validity of the covariant conservation laws for the stress–energy tensor and the spin tensor, while the second assumption enables us to expand these two currents into the δ series and discuss them in the single-pole and pole-dipole approximations. After the analysis of the independent variables and the convenient form of the covariant conservation laws, we moved on to the derivation of the equations of motion for the spinless p-brane in the pole-dipole approximation. The resulting equations of motion and boundary conditions turn out to depend on several free parameters, which have been interpreted as the effective $(p+1)$-dimensional stress–energy tensor of the brane, the p-dimensional stress–energy tensor of its boundary, and the current of the internal orbital angular momentum of the brane. We have discussed the effects of the first and second extra symmetries from the first chapter. Then we have turned to the derivation of the equations of motion and boundary conditions for the general case of a p-brane with spin. The resulting equations represent the central result of the paper. The free parameters which enter the equations are again the effective stress–energy tensors for the brane and its boundary, but now instead of the orbital angular momentum we find the total angular momentum current, which is the sum of the orbital and spin parts. In addition, the free parameters are the spin tensor currents. As an important special case, we have discussed in detail the single-pole approximation, which is characterized both by the absence of the internal orbital angular momentum and the interesting relation which constrains the possible form of the spin currents.

The third chapter contains several examples. The attention was first focused on the case of the particle in the spacetime with curvature and torsion, where the connection between the second extra symmetry and the center of mass of the particle was illustrated. Then we discussed the case of the particle with the spin in the single-pole approximation. Special attention was devoted to the example of the Dirac particle, because it turns out that the spin of the Dirac particle couples neither with its orbit nor with the spacetime curvature, in contrast to other particles. The form of the equations of motion suggests the possibility that the spin of the Dirac particle is negligible in the single-pole approximation, so we gave a short analysis of a concrete model of a Dirac field wave packet where one can see that this is indeed the case. This property has then been rewritten into the form of a covariant criterion for the comparison of the monopole spin moment with the dipole orbital moment of the particle. It was then noticed that this criterion can be applied to the non-Dirac fields. If it is satisfied, the particle must follow a geodesic, since its spin is negligible. If it is not satisfied, the orbit of the particle deviates from a geodesic, due to the interaction of the spin with the spacetime curvature and torsion.

Then the attention turned to the examples of a string, as the most interesting extended object. We have first demonstrated that the general equations of motion contain the Nambu-Goto string dynamics as a special case, together with the Neumann boundary conditions. Then we gave a short analysis of a massive rod which is at rest in spacetime, while rotating around its longitudinal axis, as a basic example of a string which has nontrivial internal angular momentum. After that our attention turned to the example of a Nambu-Goto string with two massive spinning particles attached to its ends. This example is especially interesting since it can be considered as one gross classical model of a meson, where two quarks are modeled with two spinning particles, while the QCD gluon field connecting them is modeled with a Nambu-Goto string. The equations of motion have been explicitly solved for one simple configuration of the system, after which we discussed limits of vanishing and infinite masses of the two particles, as a demonstration of the situations corresponding to the Neumann and Dirichlet boundary conditions for the Nambu-Goto string, respectively. We have also calculated the total energy and angular momentum of the system, from which we deduced the law of Regge trajectories with a correction stemming from from the internal angular momenta of two particles. The corresponding single-pole approximation was also discussed, with two Dirac particles at the end of the string, where the correction term vanishes. This is a consequence of the fact that the spin of the Dirac particle is negligible in the single-pole approximation, as was discussed in more detail in the Dirac particle case.

We finish this exposition by mentioning some open questions and further research directions. One interesting direction represents the analysis of the criteria for the comparison of the spin and orbital angular momentum, which was derived during the analysis of the Dirac particle. Starting from the question for which fields this criterion is satisfied and under which conditions, through the analysis of its form, to various speculations that it might be satisfied always and for all fundamental fields. This would mean that a strictly pointlike particle cannot have nonzero spin, regardless of the type of the field it is made of.

Besides this, various properties of complex systems may be interesting. Namely, in analogy to the last example in which the system consisted of the string and two particles attached to its ends, one can study various other configurations — a string attached to a brane, two branes connected with a string, etc. The dynamics of these kind of systems is certainly interesting from the point of view of string theory and cosmology.

Also, one can discuss interactions — a situation in which one string splits into two, or when two strings combine into one. In a similar way one can analyze the interactions of other branes, including the particle — decay, capture, scattering and other multiparticle processes.

Finally, one can study the general equations of motion for a p-brane in more detail, classify the corresponding types of matter fields which constitute the brane. This can find applications in all investigations where extended objects appear, for example in string theory, astrophysics, cosmology, etc.

Bibliography

[1] M. Mathisson, Acta. Phys. Pol. **6**, 163 (1937).

[2] J. Weyssenhoff, A. Raabe, Acta. Phys. Pol. **9**, 7 (1947).

[3] M. H. L. Pryce, Proc. R. Soc. A **195**, (62) (1948).

[4] A. Papapetrou, Proc. R. Soc. A **209**, 248 (1951).

[5] W. Tulczyjew, Acta Phys. Pol. **18**, 393 (1959).

[6] A. H. Taub, J. Math. Phys. **5**, 112 (1964).

[7] G. Dixon, Nuovo Cimento **38**, 1616 (1965); **34**, 317 (1964); Proc. R. Soc. A **314**, 499 (1970); **319**, 509 (1970); Gen. Relativ. Gravit. **4**, 199 (1973).

[8] A. Trautman, Bull. Acad. Pol. Sci., math. astr. phys. **20**, 895 (1972).

[9] F. W. Hehl, Phys. Lett. A **48**, 393 (1976).

[10] P. B. Yasskin, W. R. Stoeger, Phys. Rev. D **21**, 2081 (1980).

[11] K. Nomura, T. Shirafuji, K. Hayashi, Prog. Theor. Phys. **86**, 1239 (1991).

[12] K. Nomura, T. Shirafuji, K. Hayashi, Prog. Theor. Phys. **87**, 1275 (1992).

[13] T. Goto, Prog. Theor. Phys. **46**, 1560 (1971).

[14] Y. Nambu, Phys. Rev. D **10**, 4262 (1974).

[15] E. S. Fradkin, A. A. Tseytlin, Phys. Lett. B **158**, 316 (1985); Nucl. Phys. **B261**, 1 (1985).

[16] C. G. Callan, D. Friedan, E. J. Martinec, M. J. Perry, Nucl. Phys. **B262**, 593 (1985).

[17] T. Banks, D. Nemeschansky, A. Sen, Nucl. Phys. **B277**, 67 (1986).

[18] M. B. Green, J. H. Schwarz, E. Witten, Superstring Theory, Cambridge University Press, Cambridge, England (1987).

[19] J. Scherk, J. H. Schwarz, Phys. Lett. B **52**, 347 (1974); Nucl. Phys. **B81**, 118 (1974).

[20] T. L. Curtright, C. K. Zachos, Phys. Rev. Lett. **53**, 1799 (1984).

[21] S. Mukhi, Phys. Lett. B **162**, 345 (1985).

[22] T. Dereli, R. W. Tucker, Class. Quant. Grav. **12**, L31 (1995).

[23] B. Sazdović, Mod. Phys. Lett. A **20**, 897 (2005); Int. J. Mod. Phys. A **20**, 5501 (2005).

[24] M. Vasilić, M. Vojinović, Phys. Rev. D **73**, 124013 (2006).

[25] M. Vojinović, "Motion of the string in curved spacetime" MSc thesis, Faculty of Physics, University of Belgrade (2006).

[26] M. Vasilić, M. Vojinović, JHEP 07(2007)028.

[27] M. Blagojević, Gravitation and Gauge Symmetries, Institute of Physics Publishing, Bristol, 2001.