Renormalization in QFT

Marko Vojinović

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Abstract

This is a sketch of the renormalization procedure in perturbative QFT, based on the lecture notes I had prepared for the series of TQFTclub seminars on the same topic given in the CAMGSD group, IST, Lisbon, Portugal during the winter 2013/2014.

1   Introduction

Renormalization in Quantum Field Theory has been considered a controversial topic since its inception. There is a lot of literature on the subject, but virtually all that literature requires a nontrivial amount of knowledge of QFT itself. Moreover, in most textbooks the renormalization procedure is typically being discussed on one or more examples. Thus, for a reader who is not familiar enough with QFT itself, and who does not want to invest (a lot of) time studying QFT, the discussion of renormalization in textbooks is usually too cryptic. Moreover, as it is demonstrated on examples, the non-initiated reader has a hard time distinguishing the generic properties of renormalization idea itself from the particularities of a given example.

The aim of these lecture notes is to (hopefully) disentangle the renormalization idea from the detailed knowledge of QFT, and to present it as clearly as possible. Our approach is therefore to introduce only the absolute minimum of the QFT formalism, just enough to be able to express the renormalization procedure. The result is the abstract idea of what is being done, stripped of all particularities and details of various example QFT's which are to be renormalized. Of course, in order to illustrate these abstract ideas, one example theory is discussed in detail. However, all details regarding the example QFT — which, although important in their own right, are irrelevant for the renormalization procedure itself — will be omitted.

Note that these notes do not represent a complete account of renormalization. A lot of material is not mentioned in enough detail, or at all. Instead, the emphasis has been put on the process of "cancellation of infinities" in a given theory, and its consistency. The whole "industry" of the advanced topics — detailed analysis of renormalization group equations, beta function, scaling, self-similarity and fractals, universality, coarse graining, (ir)relevant and marginal operators, power laws and phase transitions, effective field theories, gauge theories, symmetries, anomalies, etc... — are not discussed.

Also, all the material covered by these notes can also be found in most of the usual textbooks on QFT (see for example [1, 2, 3]) in some shape and form, but not necessarily using the language and fashion that we use. In that sense, these notes do not contain any original results, but the presentation of the material contains some level of originality. If there is a textbook discussing renormalization in the way I did here, I'd like to know about it, please send me the reference!

The layout is the following. In section 2, we give a short introduction into the path integral formalism of QFT. We give a definition of the n-point functions, construct the path integral for the free field theory, and introduce the interacting QFT by means of perturbation theory. The material in this section lays minimal foundations and language of QFT necessary for the following sections. Section 3 discusses the details of the renormalization procedure, in abstract language developed in the previous section. The topics of normalization, regularization, renormalization and renormalization group are covered in turn. Section 4 deals with the explicit (and simplest possible) example of ϕ^4 scalar field theory. All steps of the

abstract procedure discussed in previous sections are explicitly demonstrated. After the normalization, regularization and renormalization of the theory, the renormalization group equations are established (to the lowest order in perturbation theory), and their meaning is discussed. At the end there is a short list of literature that cover these topics in more depth, which the reader may be interested to look at as additional sources of information.

2 Path integral formalism

2.1 The n-point functions

In the path integral formulation, a typical QFT is specified by providing a method to calculate n-point functions, as:

$$G_n(x_1, \ldots, x_n) = \int \mathcal{D}\phi \, \phi(x_1) \ldots \phi(x_n) \, e^{iS[\phi]}. \tag{1}$$

This equation is descriptive only (the so-called "statement of intent"), as we are yet to give a definition of the right-hand side. In fact, the whole topic of these notes is aimed at providing an appropriate definition of the right-hand side. We will do this through a series of plausible formal manipulations of symbols, and going out of our way to obtain a non-divergent result for all n-point functions.

The first step is to represent the n-point functions as derivatives of a suitable generating functional. Using the functional generalization of the "obvious" identity

$$f(x) \, e^{ixJ} = f\left(-i\tfrac{\partial}{\partial J}\right) e^{ixJ}, \tag{2}$$

we can formally rewrite (1) as

$$G_n(x_1, \ldots, x_n) \stackrel{\text{def}}{=} (-i)^n \frac{\delta}{\delta J(x_1)} \cdots \frac{\delta}{\delta J(x_n)} Z[J]\Big|_{J=0},$$

where the generating functional

$$Z[J] = \int \mathcal{D}\phi \, e^{iS[\phi] + i \int d^4x \, \phi(x) J(x)}$$

is yet to be defined. In this way we have reduced the problem of defining all n-point functions to the problem of defining only one object $Z[J]$. Some notes:

- The functional $S[\phi]$ is called the *classical action*, and it is assumed to be specified in advance. The functions $\phi = (\phi_1, \ldots, \phi_N)$ are fields defined over a flat Minkowski spacetime (we use the convention $(-, +, +, +)$ for the signature of the metric).

- The functions J are called the *source fields* and represent the test-functions for the whole formalism, in the sense of functional analysis. The integral of ϕJ has the whole spacetime, $\mathbb{R}^4$, as the domain of integration.

- The object $Z[J]$ goes by several names: *generating functional, partition function, path integral with sources, state sum*. We will typically use the last of those names, since it is the shortest one.

- The measure $\mathcal{D}\phi$ in the path integral is yet to be defined.

- Specifying a particular state sum $Z[J]$ is equivalent to specifying a particular quantum field theory based on the classical action $S[\phi]$.

The n-point functions are important because they can be connected to experimental measurements via the so-called LSZ formalism (constructed by H. Lehmann, K. Symanzik and W. Zimmermann). Therefore, one can intuitively understand equation (1) in the sense that the left-hand side can be "measured" in experiment, while the right-hand side can be "calculated" by the theory. In this sense one establishes a connection between a theory and experiment, and verifies the amount of agreement between the theoretical predictions and experimental data.

2.2 Perturbation theory formalism

The perturbation theory is introduced by distinguishing the *free* field theory from the *interacting* field theory. The interactions are measured by some suitable small parameter, and the interacting theory is then represented as a "perturbation" of the free theory, i.e. as an asymptotic series in the perturbation parameter. The general construction of free and interacting theories proceeds as follows. Begin by expanding the classical action functional into formal power series in the fields ϕ as:

$$\begin{aligned} S[\phi] &= S[0] + \int_{\mathbb{R}^4} d^4x\, \Delta_1(x)\phi(x) + \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^4} d^4x' \int_{\mathbb{R}^4} d^4x''\, \Delta_2(x',x'')\phi(x')\phi(x'') + \\ &\quad + \frac{1}{3!} \int_{\mathbb{R}^4} d^4x' \int_{\mathbb{R}^4} d^4x'' \int_{\mathbb{R}^4} d^4x'''\, \Delta_3(x',x'',x''')\phi(x')\phi(x'')\phi(x''') + \dots \end{aligned}$$

The series is expanded around the point $\phi = 0$, which is called the *vacuum*. The kernels Δ_1, Δ_2, ..., are differential operators of some order, and they typically contain some parameters called the *coupling constants* $g = \{g_1, g_2, \dots\}$.

When constructing the state sum $Z[J]$, one may safely omit the constant and the linear term, without loss of generality. The term quadratic in the fields is called the *kinetic* or *free* part of the action,

$$S_{\text{kin}}[\phi] \equiv \frac{1}{2} \int_{\mathbb{R}^4} d^4x' \int_{\mathbb{R}^4} d^4x''\, \Delta_2(x',x'')\phi(x')\phi(x''),$$

while cubic and higher order terms are collectively called the *interaction* part of the action,

$$S_{\text{int}}[\phi] \equiv \frac{1}{3!} \int_{\mathbb{R}^4} d^4x' \int_{\mathbb{R}^4} d^4x'' \int_{\mathbb{R}^4} d^4x'''\, \Delta_3(x',x'',x''')\phi(x')\phi(x'')\phi(x''') + \dots,$$

so we can write the classical action as

$$S[\phi] = S_{\text{kin}}[\phi] + S_{\text{int}}[\phi]. \tag{3}$$

After separating the action into kinetic and interaction parts, substitute it into the expression for the state sum, and employ again the trick (2) to rewrite it in the form

$$Z[J] = \int \mathcal{D}\phi\, e^{iS_{\text{kin}}[\phi]} e^{iS_{\text{int}}[\phi]} e^{i\int \phi J} = e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}\right]} Z_{\text{free}}[J],$$

where

$$Z_{\text{free}}[J] = \int \mathcal{D}\phi\, e^{iS_{\text{kin}}[\phi] + i\int \phi J} = \int \mathcal{D}\phi\, e^{\frac{i}{2}\int d^4x' \int d^4x''\, \phi(x')\Delta(x',x'')\phi(x'') + i\int d^4x\, \phi(x)J(x)}. \tag{4}$$

The idea here is to reduce the task of defining the path integral for an interacting QFT to the task of defining if for a free QFT. Namely, now we need to define the concept of a path integral only for the exponent which is quadratic in fields ϕ, which is a substantially simpler task.

The definition of the free-field state sum (4) is obtained by generalizing the Gaussian integral (for Euclidean QFT) and the Fresnel integral (for Minkowskian QFT) to a suitable number of dimensions. The fundamental Gauss and Fresnel integrals are

$$\int_{\mathbb{R}} dx\, e^{-x^2} = \sqrt{\pi} \qquad \text{and} \qquad \int_{\mathbb{R}} dx\, e^{ix^2} = \sqrt{\pi}\, e^{\frac{i\pi}{4}}.$$

Since we are interested in the Minkowskian QFT we will focus on the Fresnel integral. As a first step, it can be modified to contain the general quadratic form in the exponent of the integrand. By completing the square, one can easily prove that

$$\int_{\mathbb{R}} dx\, e^{\frac{i}{2}\Delta x^2 + iJx} = \sqrt{\frac{2\pi}{\Delta}}\, e^{\frac{i\pi}{4}} e^{-i\frac{J^2}{2\Delta}}, \qquad (\Delta, J \in \mathbb{R} \quad \text{and} \quad \Delta > 0). \tag{5}$$

From the point of view of QFT, this integral is "for one spacetime point". This means that the integration variable x is understood to be a function mapping "spacetime" $\mathcal{M} = \{1\}$ to a real number,

$$x : \{1\} \to \mathbb{R}.$$

The second step is to generalize (5) to n "spacetime points", $\mathcal{M} = \{1, 2, \ldots, n\}$ as:

$$\int_{\mathbb{R}^n} dx_1 \ldots dx_n \, \exp\left(\frac{i}{2} \sum_{k,l=1}^n x_k \Delta_{kl} x_l + i \sum_{k=1}^n J_k x_k \right) = \sqrt{\frac{(2\pi)^n}{\det \Delta}} e^{\frac{in\pi}{4}} \exp\left(-\frac{i}{2} \sum_{k,l=1}^n J_k (\Delta^{-1})_{kl} J_l \right), \quad (6)$$

$$(\Delta_{kl}, J_k \in \mathbb{R}, \quad \Delta = \Delta^T \quad \text{and} \quad \det \Delta > 0).$$

Here the indices $k, l, \ldots$ denote spacetime points, while integration variables $x_1, \ldots, x_n$ are functions mapping "spacetime" to real numbers,

$$x_k : \{1, 2, \ldots, n\} \to \mathbb{R}.$$

The third step is to take the limit $n \to \infty$ and to extend the spacetime from n points to a full 4-dimensional continuum manifold of points:

$$\mathcal{M} = \{1, 2, \ldots, n\} \quad \xrightarrow{n \to \infty} \quad \mathcal{M} = \mathbb{R}^4.$$

With a change of notation $k \to x \equiv (x^0, x^1, x^2, x^3)$ for a spacetime point and $x_k \to \phi(x)$ for the integration variable, the latter is now taken to be a function mapping "spacetime" to real numbers,

$$\phi : \mathbb{R}^4 \to \mathbb{R}.$$

Using a suitable change of notation from discrete to functional quantities,

$$k \to x, \qquad x_k \to \phi(x), \qquad \Delta_{kl} \to \Delta(x, x'), \qquad J_k \to J(x),$$

$$\sum_{k=1}^n \to \int_{\mathbb{R}^4} d^4 x, \qquad \int_{\mathbb{R}^n} dx_1 \ldots dx_n \to \int_{\mathbb{R}^{\mathbb{R}^4}} \mathcal{D}\phi,$$

we generalize the Fresnel integral (6) to its functional version:

$$\int_{\mathbb{R}^{\mathbb{R}^4}} \mathcal{D}\phi \, \exp\left(\frac{i}{2} \int_{\mathbb{R}^4} d^4 x' \int_{\mathbb{R}^4} d^4 x'' \, \phi(x') \Delta(x', x'') \phi(x'') + i \int_{\mathbb{R}^4} d^4 x \, \phi(x) J(x) \right) \overset{\text{def}}{=}$$

$$\overset{\text{def}}{=} \left[\lim_{n \to \infty} \sqrt{\frac{(2\pi)^n}{\det \Delta}} e^{\frac{in\pi}{4}} \right] \exp\left(-\frac{i}{2} \int_{\mathbb{R}^4} d^4 x' \int_{\mathbb{R}^4} d^4 x'' J(x') \Delta^{-1}(x', x'') J(x'') \right). \quad (7)$$

Several comments are in order:

- Equation (7) represents the definition of the left-hand side, more precisely an implicit definition of the measure $\mathcal{D}\phi$ for the path integral.

- In addition to being a definition of the path integral, equation (7) is also automatically the *evaluation* of that integral.

- The domain of integration is the set of all possible configurations (i.e. "paths") that the field ϕ can take over the spacetime $\mathbb{R}^4$. It is a set of all functions $\phi : \mathbb{R}^4 \to \mathbb{R}$, and is commonly denoted $\mathbb{R}^{\mathbb{R}^4}$.

- The operator $\Delta(x, x')$ is assumed to be regular, i.e. to have a well-defined inverse $\Delta^{-1}(x, x')$, given a suitable set of test functions $J(x)$.

- The left-hand side of (7) is precisely the expression for the free-field QFT state sum (4) which we have set out to define.

- The factor in the square brackets,

$$\mathcal{N} \equiv \lim_{n \to \infty} \sqrt{\frac{(2\pi)^n}{\det \Delta}} e^{\frac{in\pi}{4}},$$

  is called the *normalization constant*, and is highly ill-defined: in the limit $n \to \infty$ the exponent $e^{\frac{i\pi}{4}}$ oscillates, the numerator $(2\pi)^n$ diverges, while the functional determinant[1] in the denominator needs to be discussed on a case-by-case basis, depending on the properties of $\Delta(x, x')$.

This concludes the definition of the path integral in perturbative QFT. Using (4) and (7), we obtain

$$Z_{\text{free}}[J] = \mathcal{N} e^{-\frac{i}{2} \int d^4 x' \int d^4 x'' J(x') \Delta^{-1}(x', x'') J(x'')} \tag{8}$$

for the state sum of the free-field theory, and

$$Z[J] = e^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]} Z_{\text{free}}[J] \tag{9}$$

for the interacting field theory.

The interaction operator $e^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]}$ is defined as a power series expansion in the coupling constants g present in $S_{\text{int}}[\phi]$. One typically truncates the series at some particular power of g, which is called the *perturbation order*.

3 Renormalization

Despite the above efforts, the state sums (8) and (9) are still not defined quite completely. First, in most situations of physical interest, the operator $\Delta(x, x')$ is *not invertible*. Second, all n-point functions calculated from these state sums will be proportional to the ill-defined constant $\mathcal{N}$, rendering them unusable. And finally, as we shall see later, the interaction operator $e^{iS_{\text{int}}}$ introduces additional divergences into the theory, and is thus also ill-defined.

For each of these three problems there is a developed solution, and each solution entails a yet another redefinition of the theory:

Problem:	Solution:
Δ^{-1} is not defined	Wick rotation
$\mathcal{N}$ is not defined	Normalization
$e^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]}$ is not defined	Renormalization

In these notes, we will completely omit the discussion of the Wick rotation procedure. We will just state that the procedure entails a redefinition of the kinetic operator $\Delta(x, x')$ so that it becomes invertible. Its inverse will simply be denoted $\Delta^{-1}(x, x')$.

The normalization and renormalization procedures, along with the *regularization* procedure which is necessary precursor for renormalization, will be discussed in turn in the following sections.

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[1] The *functional determinant* of an operator $\Delta(x, x')$ is commonly defined by starting from the matrix identity

$$\det \Delta = e^{\text{tr} \log \Delta} = e^{\text{tr} \log[I + (\Delta - I)]},$$

and generalizing it to a functional equation. The logarithm is expanded into its Taylor series around the unit operator, so we can write explicitly

$$\det \Delta = \exp \left[\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \int_{\mathbb{R}^4} d^4 x_1 \dots \int_{\mathbb{R}^4} d^4 x_n \left[\Delta - I\right](x_1, x_2) \left[\Delta - I\right](x_2, x_3) \dots \left[\Delta - I\right](x_n, x_1) \right],$$

where

$$\left[\Delta - I\right](x, x') \equiv \Delta(x, x') - \delta^{(4)}(x - x').$$

If the spectrum of Δ is discrete, its determinant reduces to the product of its eigenvalues.

3.1   Normalization

Normalization is the procedure of defining a new and better state sum $Z_{\text{norm}}[J]$ from the old state sum $Z[J]$ given by (9), in order to eliminate the normalization constant $\mathcal{N}$ from the theory. The procedure is quite simple, and mimics the procedure of normalizing a vector, where one defines a "unit" vector by dividing the original vector by its norm. Using that analogy as motivation, we define:

$$Z[J] \xrightarrow{\text{normalization}} Z_{\text{norm}}[J] \overset{\text{def}}{=} \frac{Z[J]}{Z[J=0]} = \frac{e^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]} Z_{\text{free}}[J]}{e^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]} Z_{\text{free}}[J]\Big|_{J=0}}. \tag{10}$$

The normalization constant $\mathcal{N}$ is automatically canceled between the numerator and denominator, and is absent from $Z_{\text{norm}}[J]$.

When dealing with a free-field theory, the normalization step is enough to make the theory completely well-defined. In an interacting theory, however, the interaction operator introduces additional divergences, which can generally be divided into:

- *bubble* divergences, which appear inside J-independent terms of the interaction operator, and

- *loop* divergences, which appear inside J-dependent terms.

As we shall demonstrate later in an example, the normalization automatically removes all bubble divergences, thereby improving interacting theories even beyond the cancellation of $\mathcal{N}$. Nevertheless, the loop divergences are *not* removed by normalization, and require an additional redefinition of the theory — renormalization.

3.2   Regularization

As a necessary precursor to renormalization, we need to perform a *parametrization* of all loop divergences that appear in the interaction operator $e^{iS_{\text{int}}[-i\frac{\delta}{\delta J}]}$. This parametrization procedure is called *regularization*, and it consists of substituting the normalized state sum with the regularized state sum, which depends on the *regularization parameter*, typically denoted as ε:

$$Z_{\text{norm}}[J] \xrightarrow{\text{regularization}} Z_{\text{reg}}[J, \varepsilon].$$

In particular, this is done by defining a new interaction operator,

$$e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}\right]} \xrightarrow{\text{regularization}} e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right]}, \tag{11}$$

such that the following two requirements are satisfied:

1. In the limit $\varepsilon \to 0$ the interaction operator should reduce to the original operator (with divergences),

$$\lim_{\varepsilon \to 0} e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right]} = e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}\right]}. \tag{12}$$

2. For any fixed nonzero value of the regularization parameter ε, the *regularized state sum* must be finite, for all choices of the test function $J(x)$:

$$Z_{\text{reg}}[J, \varepsilon] \overset{\text{def}}{=} \frac{e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right]} Z_{\text{free}}[J]}{e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right]} Z_{\text{free}}[J]\Big|_{J=0}} < \infty, \qquad \text{for fixed} \quad \varepsilon \neq 0. \tag{13}$$

In general, the regularized state sum can be represented as a sum of two parts,

$$Z_{\text{reg}}[J, \varepsilon] = Z_{\text{reg}}^{\text{div}}[J, \varepsilon] + Z_{\text{reg}}^{\text{conv}}[J, \varepsilon], \qquad (\varepsilon \to 0). \tag{14}$$

The divergent part typically behaves as $\mathcal{O}(\varepsilon^{-n})$ or $\mathcal{O}(\log \varepsilon)$ for some $n \in \mathbb{N}$ as $\varepsilon \to 0$. The convergent part behaves as $\mathcal{O}(1)$ or $\mathcal{O}(\varepsilon^n)$ in the same limit.

The regularization procedure is *not unique*. In particular, the regularization parameter ε can be introduced in a variety of ways, and as a consequence the convergent piece of the regularized state sum may be different. This is a consequence of the general fact that

$$\infty = \infty + const.$$

Any particular implementation of regularization is called a *regularization scheme*, and we can differentiate between state sums $Z_{\mathrm{reg}}^{(A)}[J, \varepsilon]$ and $Z_{\mathrm{reg}}^{(B)}[J, \varepsilon]$ obtained using regularization schemes A and B, respectively. In practice, there are several well-developed techniques: cutoff, dimensional, lattice, Pauli-Villars, point-splitting, ζ-function, etc. regularizations. Each technique can be advantageous for a particular purpose, although they are conceptually all the same — they all parametrize infinite terms in the normalized state sum. We will demonstrate dimensional and cutoff regularization techniques in an example later in the text.

It should be stressed that, as long as requirements (12) and (13) are met, the choice of regularization scheme is limited only by our imagination.

3.3 Renormalization

The regularized state sum suffers from two fundamental problems. First, the state sum $Z_{\mathrm{reg}}^{(A)}[J, \varepsilon]$ (and consequently all n-point functions calculated from it) depends on the choice of the regularization scheme A, which is arbitrary. This arbitrariness needs to be addressed somehow if we are to construct a unique quantum theory. Second, all n-point functions calculated from $Z_{\mathrm{reg}}^{(A)}[J, \varepsilon]$ explicitly depend on the value of the regularization parameter ε. As this parameter is unphysical and never observed in experiment, it needs to be removed from the theory in some way.

The first problem is addressed by the so-called *renormalization group*, discussed in the next section. The second problem is addressed by renormalization.

Renormalization represents yet another (and final) redefinition of the state sum, which removes the regularization parameter from the state sum:

$$Z_{\mathrm{reg}}[J, \varepsilon] \xrightarrow{\text{renormalization}} Z_{\mathrm{ren}}[J].$$

This is done by redefining the interaction operator such that the divergent piece of the state sum is suitably removed (subtracted), and subsequently taking the limit $\varepsilon \to 0$ on the remainder. This will eventually give a finite result, independent of ε.

The interaction operator is being redefined by adding new terms to the classical action — the so-called *counterterms*:

$$e^{iS_{\mathrm{int}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right]} \xrightarrow{\text{renormalization}} e^{iS_{\mathrm{int}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right] + iS_{\mathrm{ct}}\left[-i\frac{\delta}{\delta J}, \varepsilon\right]}. \tag{15}$$

Counterterms enter the action with their own set of coupling constants $c = \{c_1, \ldots, c_K\}$. These constants are assumed to have an ε-dependent divergent piece, and an additional finite piece:

$$c = c_{\mathrm{div}}(\varepsilon) + c_{\mathrm{conv}}.$$

The divergent piece will be uniquely determined below, while the finite piece can be chosen arbitrarily, and its choice is called the *renormalization scheme*. Similar to regularization schemes, there are infinitely many renormalization schemes, and some typically used are the *on-shell* scheme, the *minimal-subtraction* scheme and the *modified minimal-subtraction* scheme. The arbitrariness of the renormalization scheme is again a manifestation of the general fact

$$\infty = \infty + const.$$

Given the framework of perturbation theory, the exponent of the counterterm action is also expanded into power series for the coupling constants, and the series is truncated at a given perturbation order.

As a consequence, the state sum is modified from the original regularized state sum to the one that has additional contribution from the counterterms:

$$Z_{\text{reg}}[J, \varepsilon] \xrightarrow{\text{renormalization}} Z_{\text{reg}}[J, \varepsilon] + Z_{\text{ct}}[J, \varepsilon], \tag{16}$$

and the counterterm state sum also splits into the divergent and finite part,

$$Z_{\text{ct}}[J, \varepsilon] = Z_{\text{ct}}^{\text{div}}[J, \varepsilon] + Z_{\text{ct}}^{\text{conv}}[J, \varepsilon], \qquad (\varepsilon \to 0). \tag{17}$$

Next, the functional form of the counterterms is chosen such that the dependence of $Z_{\text{ct}}^{\text{div}}[J, \varepsilon]$ on the sources $J(x)$ is identical to the dependence of $Z_{\text{reg}}^{\text{div}}[J, \varepsilon]$. This enables us to choose the divergent part of the counterterm coupling constants $c_{\text{div}}(\varepsilon)$ such that the requirement

$$Z_{\text{reg}}^{\text{div}}[J, \varepsilon] + Z_{\text{ct}}^{\text{div}}[J, \varepsilon] = 0 \tag{18}$$

is satisfied for all $J(x)$. This requirement basically eliminates the divergent piece from the state sum, so that we can finally define the renormalized state sum as:

$$Z_{\text{reg}}[J, \varepsilon] \xrightarrow{\text{renormalization}} Z_{\text{ren}}[J] \overset{\text{def}}{=} \lim_{\varepsilon \to 0} \left(Z_{\text{reg}}[J, \varepsilon] + Z_{\text{ct}}[J, \varepsilon] \right). \tag{19}$$

Comments:

- Definition (19), along with the requirement (18), represents the final form of the state sum for a QFT, one that is completely finite and well-defined.

- All n-point functions that are calculated from $Z_{\text{ren}}[J]$ are also finite, independent of the regularization parameter and can be compared to experiment.

- The arbitrariness in the choice of the regularization scheme (i.e. the arbitrary finite piece of $Z_{\text{reg}}^{\text{conv}}[J, \varepsilon]$ from (14)) and the arbitrariness in the choice of the renormalization scheme (i.e. the arbitrary finite piece of $Z_{\text{ct}}^{\text{conv}}[J, \varepsilon]$ from (17)) will give rise to a set of arbitrary parameters in the theory. As a consequence, (19) is *not unique*, but represents a family of theories. The discussion of this nonuniqueness is the topic of the *renormalization group*, taken in next section.

3.4 Renormalization group

Let us sum up all the steps in the construction of $Z_{\text{ren}}[J]$ so far. Starting from a classical action $S[\phi]$, equation (3), we have first constructed a "naive" state sum $Z[J]$, equation (9), which was plagued by divergences and ill-defined. Some of these problems were removed by introducing the normalized state sum $Z_{\text{norm}}[J]$, equation (10). However, the loop divergences survived, which required us to introduce a regularized state sum $Z_{\text{reg}}[J, \varepsilon]$, equations (11), (12) and (13). In order to remove the regularization parameter ε, we have introduced the counterterms into the classical theory, equation (15), and used them to cancel away the divergent parts of $Z_{\text{reg}}[J, \varepsilon]$ before taking the limit $\varepsilon \to 0$ and defining the final, renormalized state sum $Z_{\text{ren}}[J]$, equation (19).

Unfortunately, the regularization and renormalization steps were not unique, and the resulting QFT in general depends on the choices of regularization and renormalization schemes. This nonuniqueness means that we have obtained not one, but a *family* of renormalized state sums, for a given classical

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action. The whole process can be described by the following diagram:

$$S[\phi, g] \xrightarrow[\text{Wick rotation}]{\text{quantization}} Z[J, g] \xrightarrow{\text{normalization}} Z_{\text{norm}}[J, g] \xrightarrow{\text{regularization}} Z_{\text{reg}}^{(A)}[J, g, \varepsilon] \xrightarrow{\text{renormalization}} Z_{\text{ren}}^{(A)}[J, g, c_1]$$

$$Z_{\text{ren}}^{(A)}[J, g, c_2]$$

$$\vdots$$

$$Z_{\text{reg}}^{(B)}[J, g, \varepsilon] \longrightarrow Z_{\text{ren}}^{(B)}[J, g, c_3]$$

$$Z_{\text{ren}}^{(B)}[J, g, c_4]$$

$$\vdots \qquad\qquad \vdots$$

In order for the whole process to be consistent, we must require that the resulting QFT be independent of the choices of the regularization and renormalization schemes. In other words, we must prove that all possible renormalized state sums obtainable by the above procedure form a suitable *equivalence class*, thereby providing a unique QFT. More precisely, we must enforce the following requirement:

$$\forall g^n \text{ (perturbation order)}, \qquad \forall A, B \text{ (regularization schemes)}, \qquad \forall c, \tilde{c} \text{ (renormalization schemes)},$$

$$\exists g, \tilde{g} \text{ (coupling constants)}, \qquad \forall J(x) \text{ (test functions)}$$

$$Z_{\text{ren}}^{(A)}[J, g, c] = Z_{\text{ren}}^{(B)}[J, \tilde{g}, \tilde{c}]. \tag{20}$$

The equations (20) are called *renormalization group equations*. If they are satisfied, then different choices of regularization and renormalization schemes lead merely to different representatives of the same equivalence class, which represents a unique QFT.

In practice, the renormalization group equations are solved by assuming that the coupling constants g are not constants, but rather functions of the counterterm coupling constants c, $g = g(c)$. One then formulates the renormalization group equations by taking derivatives of (20) with respect to counterterm coupling constants, and enforcing appropriate boundary conditions:

$$\frac{\partial}{\partial c_k} Z_{\text{ren}}^{(A)}[J, g(c), c] + \frac{\partial g_m}{\partial c_k} \frac{\partial}{\partial g_m} Z_{\text{ren}}^{(A)}[J, g(c), c] = 0, \qquad \text{and} \qquad g(\tilde{c}) = \tilde{g}. \tag{21}$$

Comments:

- Equations (21) are a system of linear partial differential equations for unknown functions $g(c)$.

- The boundary condition $g(\tilde{c}) = \tilde{g}$ should fix a unique solution for $g(c)$. It says that the solution must contain the point $(\tilde{g}, \tilde{c})$, which is called the *kinematical point*, and is measured experimentally for a given QFT.

- The factors $\frac{\partial g_m}{\partial c_k}$ in (21) are called *beta functions*, and contain important information about various properties of the given theory.

- Historically, equations (21) were first discussed in some special cases in scalar field theory and quantum electrodynamics, and in older literature go by the name *Callan-Symanzik equations*.

- The solutions of (21) are sometimes graphically denoted on the so-called *renormalization group flow diagram*, which (in a suitably complicated theory) may look like this:

The choice of the kinematical point singles out one particular solution.

- The fact that coupling constants change their value depending on the choice of the counterterm constants c is called *running of the coupling constants.*

- In most situations, one of the counterterm coupling constants c can be traded for the so-called *scale parameter.* This establishes the dependence of coupling constants g on the length/energy scale at which measurements are being performed. Experiments at various scales may start their measurements from various kinematical points, but they must all lie on the same curve on the flow diagram.

Let us denote the number of coupling constants g as n_g and the number of counterterm coupling constants c as n_c. The number n_g is determined by the form of the classical action $S[\phi]$, while n_c is determined by the number of different types of divergences in $Z_{\mathrm{norm}}[J]$. Given this, equations (21) represent a system of n_c equations for n_g unknown functions, and if $n_g < n_c$ the system might be overdetermined, and have no solutions. In that situation the renormalization group equations *do not exist*, and the family of renormalized state sums *does not form an equivalence class*. In that sense, we have the following important characterization.

Characterization

Given a classical action $S[\phi]$ and the family of all renormalized state sums $Z_{\mathrm{ren}}[J]$:

- if the renormalization group equations (20) *do exist*, the classical action $S[\phi]$ is said to be **renormalizable**;

- if the renormalization group equations (20) *do not exist*, the classical action $S[\phi]$ is said to be **nonrenormalizable**.

The most celebrated example of a renormalizable theory is the *Standard Model of elementary particles*, while the most celebrated example of a nonrenormalizable theory is the *theory of General Relativity*.

This concludes our exposition of the renormalization. Note that there is a whole industry of further results, essentially based on the analysis of renormalization group equations and their consequences for the theory and experiment. In these notes we will not go any further into those topics. Instead, the next section will focus on a simple explicit example, aimed to illustrate the abstract procedure described so far.

4   Example: ϕ^4 theory

The theory of one real scalar field with the quartic potential is arguably the simplest nontrivial example of a renormalizable theory. In what follows, we will use it to demonstrate the renormalization procedure.

4.1   Classical theory and quantization

The classical action for the ϕ^4 theory is

$$S[\phi, g] = \int_{\mathbb{R}^4} d^4x \left[\frac{Z^2}{2} \phi(x) \left(\Box - m^2 \right) \phi(x) + \frac{\lambda}{4!} \phi^4(x) \right],$$

where we identify the set of coupling constants as $g = \{Z, m, \lambda\}$. The kinetic and potential parts of the action can be written in the form

$$S_{\text{kin}}[\phi] = \frac{1}{2} \int_{\mathbb{R}^4} d^4x' \int_{\mathbb{R}^4} d^4x'' \ \phi(x') \Delta(x', x'') \phi(x''), \qquad S_{\text{int}}[\phi] = \frac{\lambda}{4!} \int_{\mathbb{R}^4} d^4x \ \phi^4(x),$$

where the operator Δ in the kinetic term is given as

$$\Delta(x, y) = Z^2 \delta^{(4)}(x - y) \left(\Box_y - m^2 \right).$$

Note that this operator is singular, i.e. it does not have an inverse, due to the non-positive-definite Minkowski metric inside the D'Alambertian operator $\Box$. The standard procedure in these circumstances is to resort to the *Wick rotation*, which modifies the operator Δ slightly, rendering it invertible. Without getting into the discussion how this is done, for the purposes of these lecture notes we simply specify the respective inverse operator, by definition, to be

$$\Delta^{-1}(x, y) \overset{\text{def}}{=} \frac{1}{Z^2} \lim_{\xi \to 0} \int_{\mathbb{R}^4} d^4p \frac{e^{i(x-y)p}}{-p^2 - m^2 + i\xi}. \tag{22}$$

This inverse is called the *Feynman propagator*. It has the following three important properties:

1. it is symmetric,
$$\Delta^{-1}(x, y) = \Delta^{-1}(y, x),$$

2. in the limit $x \to y$ its asymptotic behavior is
$$\Delta^{-1}(x, y) \approx \frac{const}{|(x - y)^2|},$$

3. the diagonal elements are divergent (as a consequence of the above asymptotics)
$$\Delta^{-1}(x, x) = \infty. \tag{23}$$

The interaction part of the action gives rise to the interaction operator,

$$e^{iS_{\text{int}}\left[-i\frac{\delta}{\delta J}\right]} = e^{i\frac{\lambda}{4!} \int_{\mathbb{R}^4} d^4x \frac{\delta^4}{\delta J(x)^4}} = 1 + i\frac{\lambda}{4!} \int_{\mathbb{R}^4} d^4x \frac{\delta^4}{\delta J(x)^4} + \mathcal{O}(\lambda^2), \tag{24}$$

where we have expanded the exponential into power series and truncated it at the lowest nontrivial order in the interaction coupling constant λ, in line with the idea of perturbation theory. Throughout these lecture notes, we will systematically discuss the ϕ^4 theory only up to this perturbation order, since it represents the simplest possible example.

At this point we have all the necessary ingredients to construct the naive QFT state sum (9). Acting explicitly with the interaction operator (24) onto the free-field state sum $Z_{\text{free}}[J]$, we obtain the following interacting state sum:

$$Z[J] = \mathcal{N}\left[1 + \frac{1}{8}i\lambda\int d^4x[-i\Delta^{-1}(x,x)]^2 + \frac{1}{4}i\lambda\int d^4x[-i\Delta^{-1}(x,x)]\left(\int d^4x'J(x')[-i\Delta^{-1}(x,x')]\right)^2\right.$$

$$\left. + \frac{1}{24}i\lambda\int d^4x\left(\int d^4x'J(x')[-i\Delta^{-1}(x,x')]\right)^4\right]e^{-\frac{i}{2}\int d^4x'\int d^4x''J(x')\Delta^{-1}(x',x'')J(x'')}. \tag{25}$$

At this point it will prove very convenient to introduce the *Feynman rules*, which provide a graphical representation for the various expressions appearing in (25):

$$-i\Delta^{-1}(x,y) \equiv \underset{x \quad y}{\bullet\!\!-\!\!\!-\!\!\bullet}\,,$$
$$\int_{\mathbb{R}^4} d^4x\, J(x) \equiv \underset{x}{*\!\!-\!\!\!-\!\!\bullet}\,,$$
$$i\lambda\int_{\mathbb{R}^4} d^4x \equiv \underset{x}{\bm{\times}}\,.$$

Note that in the third rule, all four "legs" of the diagram are assumed to be contracted into the single point x. We draw a cross instead of a single point simply to make its four-valent structure graphically explicit. The Feynman rules establish a 1-to-1 correspondence between the analytic expressions in (25) and the *Feynman diagrams*, which we will use systematically from now on. Using Feynman rules, the state sum (25) can be rewritten as

$$Z[J] = \mathcal{N}\left[1 + \frac{1}{8}\,\text{⬥} + \frac{1}{4}\,*\!\!-\!\!\bigcirc\!\!-\!\!* + \frac{1}{24}\,\text{⨉}\right]e^{\frac{1}{2}*\!\!-\!\!*}.$$

In this graphical representation, it is easy to notice that the state sum features two potentially divergent diagrams — one source-independent *bubble* diagram, and one source-dependent *loop* diagram. In higher perturbation orders, additional more complicated diagrams of both types will appear.

4.2 Normalization

The next step in the procedure is to construct the normalized state sum (10), in order to get rid of the ill-defined normalization constant $\mathcal{N}$:

$$Z_{\text{norm}}[J] \overset{\text{def}}{=} \frac{Z[J]}{Z[J=0]} = \frac{\mathcal{N}\left[1 + \frac{1}{8}\,\text{⬥} + \frac{1}{4}\,*\!\!-\!\!\bigcirc\!\!-\!\!* + \frac{1}{24}\,\text{⨉}\right]e^{\frac{1}{2}*\!\!-\!\!*}}{\mathcal{N}\left[1 + \frac{1}{8}\,\text{⬥}\right]}.$$

We see that the normalization constant cancels away. In addition, in line with the perturbation theory, we need to re-expand the expression into power series in λ, since it appears inside the bubble diagram in the denominator. Using the Taylor expansion rule

$$\frac{1 + a\lambda}{1 + b\lambda} \approx 1 + (a-b)\lambda + \mathcal{O}(\lambda^2),$$

we see that the bubble diagrams from the numerator and the denominator will also cancel away. The cancellation of bubble diagrams is a generic property of the normalization step, and it can be proved that it happens automatically up to any order in perturbation theory. The final form of the normalized state sum is thus

$$Z_{\text{norm}}[J] = \left[1 + \frac{1}{4}\,*\!\!-\!\!\bigcirc\!\!-\!\!* + \frac{1}{24}\,\text{⨉}\right]e^{\frac{1}{2}*\!\!-\!\!*}.$$

4.3 Regularization

The normalized state sum features one loop diagram, which contains a propagator evaluated at a diagonal point. As a consequence, according to (23) this diagram will diverge, for any nontrivial value of the sources $J(x)$. The presence of this divergence requires the regularization procedure. As we have discussed previously, the regularization procedure is not unique, and in order to demonstrate this we will regularize $Z_{\text{norm}}[J]$ using two different regularization schemes.

A regularization scheme is any particular recipe to parametrize the divergence present in the state sum. We will first demonstrate the *cutoff regularization scheme*. Begin by noting that, according to (22), the diagonal value of the propagator $\Delta^{-1}(x, x)$ does not actually depend on the point x. Consequently we will denote it simply as Δ^{-1}. The first step of the cutoff regularization is to rewrite the divergent term Δ^{-1} into a spherically-symmetric form,

$$\Delta^{-1} = \frac{1}{Z^2} \lim_{\xi \to 0} \int_{\mathbb{R}^4} d^4 p \frac{1}{-p^2 - m^2 + i\xi} = \begin{pmatrix} \text{change vars to} \\ \text{4D spherical} \\ \text{coordinates} \end{pmatrix} = \frac{1}{Z^2} \lim_{\xi \to 0} \int_{S^3} d^3\Omega \int_0^\infty dp_r \frac{|I(p_r, \Omega)|}{-p_r^2 - m^2 + i\xi}.$$

The divergence is now contained in the radial integral, while the integral over the sphere S^3 is finite. We now regularize the radial integral by restricting the domain of integration — we introduce a cutoff parameter Λ, and integrate only in the domain $[0, \Lambda]$, instead of $[0, \infty)$:

$$\Delta^{-1}(\Lambda) \overset{\text{def}}{=} \frac{1}{Z^2} \lim_{\xi \to 0} \int_{S^3} d^3\Omega \int_0^\Lambda dp_r \frac{|I(p_r, \Omega)|}{-p_r^2 - m^2 + i\xi} = \cdots = \frac{i\pi^2 m^2}{Z^2} \left[\log\left(1 + \frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{m^2} \right].$$

This intervention renders the integral finite, and it can be evaluated to give the expression on the right-hand side (we omit the actual details of evaluation). The regularized propagator $\Delta^{-1}(\Lambda)$ will coincide with the original Δ^{-1} in the limit $\Lambda \to \infty$. Note that in higher perturbation orders one would have several different divergent loop diagrams, and one is supposed to regularize each of them consistently, i.e. using *the same technique* of rewriting in spherical coordinates and introducing the cutoff Λ in the radial integral.

Alternatively, we can use the *dimensional regularization scheme*. In contrast to the cutoff regularization, we will not restrict the integration domain. Instead, we will analyze the following family of integrals in D spacetime dimensions:

$$\Delta^{-1}(D, n) \overset{\text{def}}{=} \frac{1}{Z^2} \lim_{\xi \to 0} \int_{\mathbb{R}^D} d^D p \frac{1}{(-p^2 - m^2 + i\xi)^n} = \cdots = \frac{i(-1)^n \pi^{\frac{D}{2}}}{Z^2 m^{2n - D}} \frac{\Gamma\left(n - \frac{D}{2}\right)}{\Gamma(n)}.$$

Again we skip the details of evaluation, noting only that the result on the right-hand side is valid for $D < 2n$. The propagator Δ^{-1} we are interested in corresponds to the choice $D = 4$, $n = 1$, which is clearly outside of the domain of validity for the above evaluation. Moreover, that choice of the parameters puts us precisely into the pole of the Gamma function, stressing the divergent nature of Δ^{-1}. The dimensional regularization now consists of two steps. First, we perform *analytic continuation* of $\Delta^{-1}(D, n)$ to the full complex plane $D \in \mathbb{C}$ and $n \in \mathbb{C}$, except for the poles of the Gamma functions. Second, we introduce a regularization parameter ε by evaluating the analytically continued expression at the point $n = 1$ and $D = 4 - \varepsilon$. Using the asymptotic expansion of the Gamma function in the neighborhood of a pole, we obtain the following regularized value of the propagator:

$$\Delta^{-1}(\varepsilon) \equiv \Delta^{-1}(4 - \varepsilon, 1) = \frac{i\pi^2 m^2}{Z^2} \left[\frac{2}{(m\sqrt{\pi})^\varepsilon} \frac{1}{\varepsilon} + \frac{1 - \gamma}{(m\sqrt{\pi})^\varepsilon} + \mathcal{O}(\varepsilon) \right], \qquad (\varepsilon \to 0).$$

Here $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant. In the limit $\varepsilon \to 0$ this reduces to the wanted propagator Δ^{-1}. Again, note that in higher perturbation orders all divergent integrals should systematically be regularized by this same technique of analytic continuation to $D = 4 - \varepsilon$ dimensions of spacetime.

We can now see that, in both the cutoff and dimensional regularization scheme, the diagonal value of the propagator has the generic structure

$$\Delta^{-1} = \Delta_{\text{div}}^{-1} + \Delta_{\text{conv}}^{-1}. \tag{26}$$

The difference between the two regularizations consists of the way the regularization parameter has been introduced, and in the value of the finite piece $\Delta_{\mathrm{conv}}^{-1}$. Namely, equating the results for Δ^{-1} in both regularization schemes, we obtain

$$\log\left(1+\frac{\Lambda^2}{m^2}\right) - \frac{\Lambda^2}{m^2} = \frac{2}{(m\sqrt{\pi})^\varepsilon}\frac{1}{\varepsilon} + \frac{1-\gamma}{(m\sqrt{\pi})^\varepsilon} + \mathcal{O}(\varepsilon).$$

This equation establishes the functional relationship between the parameters Λ and ε. In addition, in the limit $\Lambda \to \infty$, $\varepsilon \to 0$ it represents an explicit example of the general statement

$$\infty = \infty + const.$$

Now that we have performed the parametrization of the infinity of Δ^{-1}, we can explicitly construct the regularized state sum (13), noting that we have satisfied the requirement (12). Proceeding with the choice of dimensional regularization scheme, and using (26), we can write the regularized state sum in the form

$$Z_{\mathrm{reg}}^{(\mathrm{dr})}[J, g, \varepsilon] = Z_{\mathrm{reg}}^{\mathrm{div}}[J, g, \varepsilon] + Z_{\mathrm{reg}}^{\mathrm{conv}}[J, g, \varepsilon],$$

where

$$Z_{\mathrm{reg}}^{\mathrm{div}}[J, \varepsilon] = \frac{1}{4}\left(\ast\!\!-\!\!\underset{}{\bigcirc}\!\!-\!\!\ast\right)_{\mathrm{div}} e^{\frac{1}{2}\ast\!\!-\!\!\ast} \tag{27}$$

and

$$Z_{\mathrm{reg}}^{\mathrm{conv}}[J, \varepsilon] = \left[1 + \frac{1}{4}\left(\ast\!\!-\!\!\underset{}{\bigcirc}\!\!-\!\!\ast\right)_{\mathrm{conv}} + \frac{1}{24}\,\includegraphics{x}\right] e^{\frac{1}{2}\ast\!\!-\!\!\ast}. \tag{28}$$

Here the divergent and convergent parts of the "pin" diagram are obtained by substituting the corresponding regularized expressions

$$\Delta_{\mathrm{div}}^{-1} = \frac{i\pi^2 m^2}{Z^2}\frac{2}{(m\sqrt{\pi})^\varepsilon}\frac{1}{\varepsilon}, \qquad \Delta_{\mathrm{conv}}^{-1} = \frac{i\pi^2 m^2}{Z^2}\frac{1-\gamma}{(m\sqrt{\pi})^\varepsilon} + \mathcal{O}(\varepsilon), \tag{29}$$

in place of the diagonal propagator in the diagram evaluation. This completes the construction of the regularized state sum.

4.4 Renormalization

The final step in the construction of QFT is the renormalization. In order to eliminate the unphysical regularization parameter ε from the regularized state sum, we must first "subtract" the divergent piece, and then take the limit $\varepsilon \to 0$ of the remainder. To facilitate the subtraction of the divergent piece, we must change the definition of the classical action of the theory, by adding *counterterms*. In this particular case, we choose the action

$$S[\phi, g, c] = \int_{\mathbb{R}^4} d^4x \left[\frac{1}{2}Z^2\phi(x)\left(\Box - m^2\right)\phi(x) + \frac{\lambda}{4!}\phi^4(x) - \frac{c}{2}\phi^2(x)\right], \tag{30}$$

where the last term on the right-hand side is the counterterm. Its ϕ-dependence has been judiciously chosen to match the functional form of the divergent pin diagram, as we shall see below. Note that despite being quadratic in the field ϕ, this term is not part of the kinetic piece of the action, but rather remains in the interaction piece, giving a contribution to the interaction operator.

The counterterm coupling constant is assumed to have a divergent ε-dependent piece and the additional finite piece,

$$c = c_{\mathrm{div}}(\varepsilon) + c_{\mathrm{conv}}. \tag{31}$$

Each particular choice of the finite coupling constant amount to one particular *renormalization scheme*. The divergent piece will be used to cancel the divergent piece of the pin diagram.

At this point we should repeat the whole procedure from the very beginning with the new action[2]. The presence of the counterterm induces an additional Feynman rule,

$$ic \int_{\mathbb{R}^4} d^4x \equiv \underset{x}{\bullet}\!-\!\!\overset{}{\textcircled{c}}\!\!-\!\underset{x}{\bullet}\,.$$

Similar to the case of the λ vertex, the counterterm vertex is assumed to be contracted to a single point x, while two "legs" are depicted in the diagram only to emphasize the two-valent nature of the vertex. Using this, we first construct the new naive state sum,

$$Z[J] = \mathcal{N}\left[1 + \frac{1}{8}\,\text{\Large O}\text{\Large O} + \frac{1}{4}\,\ast\!\!-\!\!\overset{\text{\small O}}{}\!\!-\!\!\ast + \frac{1}{24}\,\overset{\ast\ \ast}{\underset{\ast\ \ast}{\times}} + \frac{1}{2}\,\ast\!\!-\!\!\textcircled{c}\!\!-\!\!\ast + \frac{1}{2}\,\text{\Large O}_{\!\!c}\right] e^{\frac{1}{2}\,\ast\!-\!\ast}.$$

It now features two additional diagrams, due to the presence of the counterterm. One of those is a bubble diagram.

Normalization of the naive state sum removes the normalization constant $\mathcal{N}$, and the two bubble diagrams:

$$Z_{\text{norm}}[J] = \left[1 + \frac{1}{4}\,\ast\!\!-\!\!\overset{\text{\small O}}{}\!\!-\!\!\ast + \frac{1}{24}\,\overset{\ast\ \ast}{\underset{\ast\ \ast}{\times}} + \frac{1}{2}\,\ast\!\!-\!\!\textcircled{c}\!\!-\!\!\ast\right] e^{\frac{1}{2}\,\ast\!-\!\ast}.$$

The regularization step is identical as before, only now we can identify the counterterm contribution to the state sum (see (16) and (17)) as

$$Z_{\text{ct}}^{\text{div}}[J,\varepsilon] = \frac{1}{2}\left(\ast\!\!-\!\!\textcircled{c}\!\!-\!\!\ast\right)_{\text{div}} e^{\frac{1}{2}\,\ast\!-\!\ast}, \qquad Z_{\text{ct}}^{\text{conv}}[J,\varepsilon] = \frac{1}{2}\left(\ast\!\!-\!\!\textcircled{c}\!\!-\!\!\ast\right)_{\text{conv}} e^{\frac{1}{2}\,\ast\!-\!\ast}.$$

Using this and (27), we can rewrite the renormalization condition (18) as

$$\frac{1}{4}\left(\ast\!\!-\!\!\overset{\text{\small O}}{}\!\!-\!\!\ast\right)_{\text{div}} + \frac{1}{2}\left(\ast\!\!-\!\!\textcircled{c}\!\!-\!\!\ast\right)_{\text{div}} = 0. \tag{32}$$

Note that both terms have identical functional dependence on sources $J(x)$. Explicitly,

$$\frac{1}{4}\left(\ast\!\!-\!\!\overset{\text{\small O}}{}\!\!-\!\!\ast\right)_{\text{div}} =$$

$$= \frac{1}{4}i\lambda \int d^4x \int d^4x'\, J(x') \int d^4x''\, J(x'')\left[-i\Delta^{-1}(x,x')\right]\left[-i\Delta^{-1}(x,x'')\right]\left[-i\Delta^{-1}(x,x)\right]_{\text{div}}$$

$$= \frac{\lambda}{4}\Delta_{\text{div}}^{-1}\int d^4x \left(\int d^4x'\, J(x')\left[-i\Delta^{-1}(x,x')\right]\right)^2,$$

$$\frac{1}{2}\left(\ast\!\!-\!\!\textcircled{c}\!\!-\!\!\ast\right)_{\text{div}} =$$

$$= \frac{1}{2}ic_{\text{div}}\int d^4x \int d^4x'\, J(x') \int d^4x''\, J(x'')\left[-i\Delta^{-1}(x,x')\right]\left[-i\Delta^{-1}(x,x'')\right]$$

$$= \frac{i}{2}c_{\text{div}}\int d^4x \left(\int d^4x'\, J(x')\left[-i\Delta^{-1}(x,x')\right]\right)^2.$$

In fact, the counterterm in (30) has been chosen precisely to make this happen. Consequently, the renormalization condition can be satisfied identically with the following choice of the divergent part for the counterterm coupling constant:

$$c_{\text{div}}(\varepsilon) = \frac{i\lambda}{2}\Delta_{\text{div}}^{-1} = -\frac{\lambda\pi^2 m^2}{Z^2}\frac{1}{(m\sqrt{\pi})^\varepsilon}\frac{1}{\varepsilon}, \qquad (\varepsilon \to 0).$$

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[2]Arguably, if we were to be fully consistent we should have started with the action (30) to begin with. Unfortunately, in practice one cannot know in advance what functional form the counterterms should have prior to the analysis of loop divergences in the action without counterterms.

Here we have used (29).

Once this has been established, we can take the limit $\varepsilon \to 0$ as per (19) to obtain the final renormalized state sum:

$$Z_{\text{ren}}^{(\text{dr})}[J, g, c_{\text{conv}}] = \left[1 + \frac{1}{4}\left(\;\text{⊶○⊷}\;\right)_{\text{conv}} + \frac{1}{2}\left(\;\text{⊶©⊷}\;\right)_{\text{conv}} + \frac{1}{24}\;\text{⧓}\;\right] e^{\frac{1}{2}\,\text{⊶⊷}}. \qquad (33)$$

This state sum is completely free both of the divergence and of the regularization parameter. All n-point functions calculated from this state sum are finite. Nevertheless, the state sum still depends on the choices of regularization and renormalization schemes. This dependence is manifested through the choice of the finite part of Δ^{-1} in the pin diagram, and the choice of the finite counterterm coupling constant c_{conv} in the counterterm diagram. It is important to note that these two finite parts sum up into a single free parameter, in the same way that their divergent pieces were summed to zero in (32):

$$\frac{1}{4}\left(\;\text{⊶○⊷}\;\right)_{\text{conv}} + \frac{1}{2}\left(\;\text{⊶©⊷}\;\right)_{\text{conv}} =$$

$$= \left(\frac{\lambda}{4}\Delta_{\text{conv}}^{-1} + \frac{i}{2}c_{\text{conv}}\right) \int d^4x \left(\int d^4x'\; J(x')\left[-i\Delta^{-1}(x, x')\right]\right)^2$$

$$= \frac{i}{2}\underbrace{\left(\frac{\lambda\pi^2 m^2}{2Z^2}(1 - \gamma) + c_{\text{conv}}\right)}_{c} \int d^4x \left(\int d^4x'\; J(x')\left[-i\Delta^{-1}(x, x')\right]\right)^2.$$

The value of the new combined parameter c is fixed by the combined choices of regularization and renormalization schemes, and it should not be confused with the original counterterm coupling constant c in (31).

In this sense, the renormalized state sum represents a one-parameter family of QFT's, where the value of the free parameter encapsulates the nonuniqueness in the choices of regularization and renormalization schemes. What is left is to demonstrate that all these QFT's are actually only different representatives of the same equivalence class.

4.5 Renormalization group equations

In order to demonstrate that the family of QFT's described by the renormalized state sum (33), we need to establish the renormalization group equations (20). The form (21) of those equations is in our case given by a single partial differential equation

$$\left(\frac{\partial}{\partial c} + \frac{\partial Z^2}{\partial c}\frac{\partial}{\partial Z^2} + \frac{\partial m^2}{\partial c}\frac{\partial}{\partial m^2} + \frac{\partial \lambda}{\partial c}\frac{\partial}{\partial \lambda}\right) Z_{\text{ren}}[J; Z^2(c), m^2(c), \lambda(c); c] = 0$$

for the three functions $Z(c)$, $\lambda(c)$ and $m(c)$, with the boundary condition specified by the kinematical point $(\tilde{Z}, \tilde{m}, \tilde{\lambda}, \tilde{c})$. This equation has the following solution (we omit the actual process of finding it):

$$Z^2(c) = \tilde{Z}^2, \qquad \lambda(c) = \tilde{\lambda}, \qquad m^2(c) = \tilde{m}^2 + \frac{(2\pi)^4}{Z^2}(\tilde{c} - c). \qquad (34)$$

In particular, the coupling constants Z and λ are constant, while m is not, and "runs" as c changes value.

The meaning of the running of m is the following. Suppose we have made two different choices for the regularization and renormalization scheme, leading to two renormalized state sums, having parameters c_1 and c_2 respectively. The sense in which the second state sum is equivalent to the first is given by equation (34) — since both theories must agree on the position of the kinematical point, we have

$$m_1^2 = \tilde{m}^2 + \frac{(2\pi)^4}{Z^2}(\tilde{c} - c_1), \qquad m_2^2 = \tilde{m}^2 + \frac{(2\pi)^4}{Z^2}(\tilde{c} - c_2).$$

Eliminating the kinematical point from these equations, we obtain

$$m_2^2 = m_1^2 + \frac{(2\pi)^4}{Z^2}(c_1 - c_2). \tag{35}$$

This means that if we choose the coupling constants m_1 and m_2 to satisfy (35), the two state sums will be exactly equal,

$$Z_{\mathrm{ren}}[J; Z, \lambda, m_1; c_1] = Z_{\mathrm{ren}}[J; Z, \lambda, m_2; c_2],$$

and consequently all n-point functions will also be equal. Intuitively, we imagine one physicist to use regularization and renormalization schemes which lead to the choice c_1, while the other physicist uses different schemes, leading to the choice c_2. The predictions of their respective theories will agree if they choose masses m_1 and m_2 respectively, such that (35) holds. As a consequence, their two apparently different QFT's are actually equivalent, and are just two different representatives of the same equivalence class of theories.

Consequently, if the classical action is *renormalizable*, the corresponding QFT is *unique*, i.e. independent of the choice of regularization and renormalization schemes used to construct a representative state sum.

To conclude the analysis of the ϕ^4 example, let us note the following. In the first perturbation order in λ the state sum had only one loop divergence, giving rise to one counterterm in the action, and consequently one parameter c in the renormalized state sum. The renormalization group equations were therefore easy to satisfy, given that we had three available coupling constants $g = \{Z, \lambda, m\}$ to "absorb" the various choices of c. This established the proof of renormalizability of ϕ^4 theory up to the first perturbation order.

However, in higher perturbation orders the state sum will feature additional loop divergences, each necessitating one counterterm, thus giving rise to multiple counterterm parameters $c = \{c_1, c_2, \dots\}$. The *full proof of renormalizability* of the ϕ^4 theory entails the proof that only *three* counterterms are enough to compensate for *all* loop divergences that can appear in the state sum, in arbitrary perturbation order. Namely, in order to make sure that renormalization group equations are not overdetermined, we are allowed to have at most three counterterm parameters $c = \{c_Z, c_m, c_\lambda\}$ as variables for the three coupling constants Z, m and λ. The full proof of renormalizability thus proceeds by analyzing the type of additional loop divergences that appear as one moves recursively from n-th to $(n+1)$-th perturbation order. In this way it can be shown that exactly three counterterms are necessary and sufficient to compensate for all loop divergences, rendering the ϕ^4 theory fully renormalizable.

References

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