TENSOR CALCULUS

Part 2: tensor analysis

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Chapter 2

Tensor analysis

2.1 Tensor fields, parallel transport

So far we have been constructing the tensor algebra from a tangent space at one point $\mathcal{P}$ on the manifold $\mathcal{M}$. Now we want to pick another point, $\mathcal{Q}$, construct an analogous tensor algebra there, and then compare tensors at those different points.

Begin with the definition of the concept of a **field**. A **tensor field** is a map from the manifold $\mathcal{M}$ into a tensor algebra T:

$$\mathcal{F} \,:\, \mathcal{M} \to T, \qquad \mathcal{P} \mapsto T(\mathcal{P}),$$

where $\mathcal{P} \in \mathcal{M}$.

Since at every point the tensor algebra has the structure

$$T(\mathcal{P}) = T_{0,0} \oplus T_{1,0} \oplus T_{0,1} \oplus T_{2,0} \oplus T_{1,1} \oplus T_{0,2} \oplus \dots,$$

the field $\mathcal{F}$ also naturally splits into a direct sum of a **scalar field**, **vector field**, **1-form field**, and all other **tensor fields**, according to the nontrivial part of its codomain. Essentially, a tensor field is specified once one has attached some tensor to every point on a manifold, each tensor being the element of appropriate tensor algebra at that point.

One can naturally ask the question how to compare the value of a, say, vector field at two different points $\mathcal{P}$ and $\mathcal{Q}$. The values of the field, although both being vectors, at the two points belong to two different and unrelated spaces, so one cannot just "subtract" them. Such operation is defined only within a tensor algebra *at one point*, not across several points.

In order to address this issue, one needs to define the idea of specifying what vectors are to be considered "the same" in two different points. This process is called **parallel transport**. In order to specify a rule for parallel trasport, take some set of basis vectors $\mathbf{e}_\mu(\mathcal{Q}) \in T\mathcal{M}_\mathcal{Q}$, and specify how do those vectors "look" when one moves them slightly into a neighboring point $\mathcal{P}$ which is separated from $\mathcal{P}$ by some infinitesimal "distance" dx^ν in the direction of the $\mathbf{e}_\nu(\mathcal{P})$ vector:

$$\mathbf{e}_\mu(\mathcal{Q}) \equiv \mathbf{e}_\mu(\mathcal{P}) + \Gamma^\lambda{}_{\mu\nu}(\mathcal{P}) dx^\nu \mathbf{e}_\lambda(\mathcal{P}), \qquad \text{where} \qquad \mathcal{Q} \equiv \mathcal{P} + dx^\nu \mathbf{e}_\nu(\mathcal{P}).$$

Note:

- In the above equation **index ν is fixed, and not summed over, despite repeating twice!**
- The "distance" parameter dx^ν is defined as a parameter of a coordinate curve joining $\mathcal{P}$ and $\mathcal{Q}$, while $\mathbf{e}_\nu(\mathcal{P})$ is the tangent vector to that curve at $\mathcal{P}$. We have not introduced yet the "real" notion of "metric distance" between points on a manifold, so this is the best we can do at the moment.
- The coefficients $\Gamma^\lambda{}_{\mu\nu}$ that define the actual correspondence between $\mathbf{e}_\mu(\mathcal{Q})$ and $\mathbf{e}_\mu(\mathcal{P})$ are called **connection coefficients**. They may be specified differently from point to point, so are therefore functions of $\mathcal{P}$.

- When one constructs the vector at $\mathcal{P}$ starting from a vector at $\mathcal{Q}$ and using connection coefficients, it is said that the **vector is parallely transported from $\mathcal{Q}$ to $\mathcal{P}$**.
- As the connection coefficients depend on index ν, they may be different when one transports a vector from $\mathcal{Q}$ to different directions. Therefore, **parallel transport depends on the path along which the vector is being transported**.
- **Connection coefficients are not components of any tensor**, despite their index-like notation. There is no geometric object in tensor algebra at a point that would correspond to these. We shall discuss this issue further below.

Once we have specified the rule for parallel transport of basis vectors $\mathbf{e}_\mu$, we can easily construct the appropriate rule for the parallel transport of basis 1-forms $\mathbf{e}^\mu(\mathcal{Q}) \in T\mathcal{M}^*_{\mathcal{Q}}$ into the point $\mathcal{P}$. This is done simply by using the requirement that the parallely transported basis of 1-forms should be biorthogonal to the paralelly transported basis of vectors. Therefore, we have:

$$\mathbf{e}^\mu(\mathcal{Q}) \equiv \mathbf{e}^\mu(\mathcal{P}) - \Gamma^\mu{}_{\lambda\nu}(\mathcal{P})dx^\nu \mathbf{e}^\lambda(\mathcal{P}), \qquad \text{where} \qquad \mathcal{Q} \equiv \mathcal{P} + dx^\nu \mathbf{e}_\nu(\mathcal{P}).$$

Note that the sign in front of the connection coefficients is changed from plus to minus.

At this point we have all the neccessary pieces to perform a parallel transport of any vector (and indeed, any tensor) by keeping its components fixed and transporting the basis. For example, if $\boldsymbol{A}(\mathcal{Q}) = A^\mu(\mathcal{Q})\mathbf{e}_\mu(\mathcal{Q})$, we have:

$$\underbrace{\boldsymbol{A}_{PT}(\mathcal{Q}\to\mathcal{P})}_{\text{transported vector}} = \underbrace{A^\mu(\mathcal{Q})}_{\text{fixed components}} \underbrace{\mathbf{e}_\mu(\mathcal{P} + dx^\nu \mathbf{e}_\nu(\mathcal{P}))}_{\text{transported basis}} = A^\mu(\mathcal{Q})\left[\mathbf{e}_\mu(\mathcal{P}) + \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})dx^\nu \mathbf{e}_\lambda(\mathcal{P})\right] =$$

$$= \underbrace{\left[A^\lambda(\mathcal{Q}) + \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})A^\mu(\mathcal{P})dx^\nu\right]}_{\text{transported components}} \mathbf{e}_\lambda(\mathcal{P}) = A^\mu_{PT}(\mathcal{P})\mathbf{e}_\mu(\mathcal{P}).$$

Note that in the third step we have substituted $A^\mu(\mathcal{Q})$ for $A^\mu(\mathcal{P})$, since $A^\mu(\mathcal{Q})dx^\nu = A^\mu(\mathcal{P})dx^\nu +$ second order differentials (which can be neglected).

Similarly, one can calculate the components of any parallely transported tensor, simply by transporting each of the vectors in its basis. Once transported to the point $\mathcal{Q}$, the vector $\boldsymbol{A}$ can then be compared to some other vector residing at that point. This is the basis we need in order to define the concept of a **derivative** of a tensor field.

2.2 Covariant and exterior derivatives, commutators

The notation of parallel transport may be somewhat cumbersome and confusing. Therefore, rewrite the rule for parallel transport of basis vectors in the form

$$\mathbf{e}_\mu(\mathcal{Q}) - \mathbf{e}_\mu(\mathcal{P}) = \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})dx^\nu \mathbf{e}_\lambda(\mathcal{P}),$$

or in the form

$$\frac{\mathbf{e}_\mu(\mathcal{Q}) - \mathbf{e}_\mu(\mathcal{P})}{dx^\nu} = \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})\mathbf{e}_\lambda(\mathcal{P}).$$

Now given that dx^ν is considered infinitesimal, and since $\mathcal{Q} \equiv \mathcal{P} + dx^\nu \mathbf{e}_\nu(\mathcal{P})$, we can take the limit $dx^\nu \to 0$ and write the above as

$$\lim_{dx^\nu \to 0} \frac{\mathbf{e}_\mu(\mathcal{P} + dx^\nu \mathbf{e}_\nu(\mathcal{P})) - \mathbf{e}_\mu(\mathcal{P})}{dx^\nu} = \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})\mathbf{e}_\lambda(\mathcal{P}).$$

The left-hand side is actually the usual definition of a **derivative**, in the direction of $\mathbf{e}_\nu$. Denote it as $\boldsymbol{\nabla}_{\mathbf{e}_\nu} \equiv \boldsymbol{\nabla}_\nu$, and write the parallel transport rule in the form:

$$\boldsymbol{\nabla}_{\mathbf{e}_\nu}\mathbf{e}_\mu(\mathcal{P}) = \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})\mathbf{e}_\lambda(\mathcal{P}).$$

Similarly, for basis 1-forms we have:

$$\boldsymbol{\nabla}_{\mathbf{e}_\nu}\mathbf{e}^\mu(\mathcal{P}) = -\Gamma^\mu{}_{\lambda\nu}(\mathcal{P})\mathbf{e}^\lambda(\mathcal{P}).$$

Now we have all the ingredients to construct a **directional derivative of a vector field**, in the following way. Let a vector field $\boldsymbol{A}(\mathcal{M})$ be specified. At point $\mathcal{Q}$ it has the value $\boldsymbol{A}(\mathcal{Q})$, while at point $\mathcal{P}$ it has the value $\boldsymbol{A}(\mathcal{P})$. We want to subtract those two vectors, take the ratio with respect to "distance" between $\mathcal{P}$ and $\mathcal{Q}$, and take the limit when this distance goes to zero. But we cannot just subtract vectors in different points. Instead, **first perform parallel transport** of the vector $\boldsymbol{A}(\mathcal{Q})$ into the point $\mathcal{P}$, and **then subtract** from it the actual value $\boldsymbol{A}(\mathcal{P})$ that the vector field has at that point. In other words,

$$\boldsymbol{\nabla}_\nu\boldsymbol{A} \equiv \lim_{dx^\nu\to 0}\frac{\boldsymbol{A}_{PT}(\mathcal{Q}\to\mathcal{P}) - \boldsymbol{A}(\mathcal{P})}{dx^\nu} = \lim_{dx^\nu\to 0}\frac{A^\mu_{PT}(\mathcal{P})\mathbf{e}_\mu(\mathcal{P}) - A^\mu(\mathcal{P})\mathbf{e}_\mu(\mathcal{P})}{dx^\nu} =$$

$$= \lim_{dx^\nu\to 0}\frac{A^\lambda(\mathcal{Q}) - A^\lambda(\mathcal{P}) + \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})A^\mu(\mathcal{P})dx^\nu}{dx^\nu}\mathbf{e}_\lambda(\mathcal{P}) = \left[\partial_\nu A^\lambda(\mathcal{P}) + \Gamma^\lambda{}_{\mu\nu}(\mathcal{P})A^\mu(\mathcal{P})\right]\mathbf{e}_\lambda(\mathcal{P}).$$

This being the definition, the rules for actual calculation of directional derivative are much more convenient when expressed using the above formulas for parallel transport of basis vectors. Pick a coordinate system at point $\mathcal{P}$ on the manifold, and express the vector field in the form $\boldsymbol{A}(x) = A^\mu(x)\mathbf{e}_\mu(x)$. Then calculate the directional derivative using the "chain rule":

$$\boldsymbol{\nabla}_\nu\boldsymbol{A}(x) = [\underbrace{\boldsymbol{\nabla}_\nu A^\mu(x)}_{\partial_\nu A^\mu(x)}]\mathbf{e}_\mu(x) + A^\mu(x)\underbrace{\boldsymbol{\nabla}_\nu\mathbf{e}_\mu(x)}_{\Gamma^\lambda{}_{\mu\nu}(x)\mathbf{e}_\lambda(x)} = \left[\partial_\nu A^\lambda(x) + \Gamma^\lambda{}_{\mu\nu}(x)A^\mu(x)\right]\mathbf{e}_\lambda(x).$$

Evaluating this at $x = x(\mathcal{P})$ we see that it is identical to the definition calculation. Note that in the first term the directional derivative acts on an ordinary function just like an ordinary derivative.

The elegance of the latter method is obvious, and we can use it to calculate the directional derivative of an **arbitrary tensor**. For example, given a tensor field $\boldsymbol{A}\in T_{2,1}$, one can calculate its directional derivative as follows:

$$\begin{aligned}
\boldsymbol{\nabla}_\rho\boldsymbol{A} &= \boldsymbol{\nabla}_\rho\left(A^{\mu\nu}{}_\lambda\mathbf{e}_\mu\otimes\mathbf{e}_\nu\otimes\mathbf{e}^\lambda\right)\\
&= \boldsymbol{\nabla}_\rho A^{\mu\nu}{}_\lambda\mathbf{e}_\mu\otimes\mathbf{e}_\nu\otimes\mathbf{e}^\lambda + A^{\mu\nu}{}_\lambda\,\boldsymbol{\nabla}_\rho\mathbf{e}_\mu\otimes\mathbf{e}_\nu\otimes\mathbf{e}^\lambda\\
&\quad + A^{\mu\nu}{}_\lambda\mathbf{e}_\mu\otimes\,\boldsymbol{\nabla}_\rho\mathbf{e}_\nu\otimes\mathbf{e}^\lambda + A^{\mu\nu}{}_\lambda\mathbf{e}_\mu\otimes\mathbf{e}_\nu\otimes\,\boldsymbol{\nabla}_\rho\mathbf{e}^\lambda\\
&= \left[\partial_\rho A^{\mu\nu}{}_\lambda + \Gamma^\mu{}_{\sigma\rho}A^{\sigma\nu}{}_\lambda + \Gamma^\nu{}_{\sigma\rho}A^{\mu\sigma}{}_\lambda - \Gamma^\sigma{}_{\lambda\rho}A^{\mu\nu}{}_\sigma\right]\mathbf{e}_\mu\otimes\mathbf{e}_\nu\otimes\mathbf{e}^\lambda
\end{aligned}$$

The resulting expression in square brackets is usually called **covariant derivative** of the components of a tensor, and denoted as:

$$\nabla_\rho A^{\mu\nu}{}_\lambda \equiv \partial_\rho A^{\mu\nu}{}_\lambda + \Gamma^\mu{}_{\sigma\rho}A^{\sigma\nu}{}_\lambda + \Gamma^\nu{}_{\sigma\rho}A^{\mu\sigma}{}_\lambda - \Gamma^\sigma{}_{\lambda\rho}A^{\mu\nu}{}_\sigma.$$

One can understand the structure of this expression in the following way:

- The first term is the ordinary derivative of a function.
- For every "up" index in $A^{\mu\nu}{}_\lambda$ add one term of the form ΓA, where this index appears as an upper one on Γ, while the corresponding one on A is contracted to a first lower index of Γ.
- For every "down" index in $A^{\mu\nu}{}_\lambda$ subtract one term of the form ΓA, where the index appears as a first lower on Γ, while the corresponding one on A is contracted to the upper index of Γ.
- The second lower index on all Γ is always the index of the differentiation.

The above construction is unique, and the procedure of "correcting indices" with connection coefficients corresponds precisely to differentiating the basis vectors and 1-forms of a tensor.

Note that from the above formula it is obvious that Γ **are not components of a tensor!** Indeed, it is easy to verify that the transformation rule for $\Gamma^{\lambda}{}_{\mu\nu}$ is:

$$\Gamma^{\lambda'}{}_{\mu'\nu'} = M^{\lambda'}{}_{\lambda} M^{\mu}{}_{\mu'} M^{\nu}{}_{\nu'} \Gamma^{\lambda}{}_{\mu\nu} + M^{\lambda'}{}_{\lambda} \partial_{\mu'} M^{\lambda}{}_{\nu'}.$$

The extra term on the right spoils the tensorial transformation properties of Γ.

So far we have constructed the directional derivative in the ν direction. The directional derivative in the direction of an arbitrary vector $\boldsymbol{B} = B^{\mu}\mathbf{e}_{\mu}$ is nothing but an appropriate linear combination:

$$\boldsymbol{\nabla}_{\boldsymbol{B}}\boldsymbol{A} = B^{\nu}\, \boldsymbol{\nabla}_{\nu}\boldsymbol{A}.$$

However, sometimes it is convenient to consider a **derivative without specifying any direction**. Here basis 1-forms offer themselves as a very useful tool. Define the **gradient** $\boldsymbol{\nabla}$ of a tensor field $\boldsymbol{A}$ as:

$$\boldsymbol{\nabla}\boldsymbol{A} \equiv \mathbf{e}^{\mu} \otimes \boldsymbol{\nabla}_{\mu}\boldsymbol{A}.$$

Recover the directional derivative in the direction of the vector $\boldsymbol{B}$ by letting the basis 1-forms act on $\boldsymbol{B}$, ie. construct an inner product of the gradient with the vector $\boldsymbol{B}$:

$$\Big(\boldsymbol{\nabla}\boldsymbol{A}\Big)\cdot\boldsymbol{B} = C\left(\Big(\boldsymbol{\nabla}\boldsymbol{A}\Big)\otimes\boldsymbol{B}\right) = \mathbf{e}^{\mu}[\boldsymbol{B}]\otimes\boldsymbol{\nabla}_{\mu}\boldsymbol{A} = B^{\mu}\,\boldsymbol{\nabla}_{\mu}\boldsymbol{A} = \boldsymbol{\nabla}_{\boldsymbol{B}}\boldsymbol{A}.$$

In this sense, the gradient operator, $\boldsymbol{\nabla} \equiv \mathbf{e}^{\mu}\otimes\boldsymbol{\nabla}_{\mu}$ is the **most general derivative operator** of any tensor, and every other differential operator may be constructed from it. Its action can be summarized as the action on an ordinary function plus the action on basis vectors and basis 1-forms:

$$\boldsymbol{\nabla} f = (\partial_{\mu} f)\mathbf{e}^{\mu}, \qquad \boldsymbol{\nabla}\mathbf{e}_{\mu} = \Gamma^{\lambda}{}_{\mu\nu}\mathbf{e}^{\nu}\otimes\mathbf{e}_{\lambda}, \qquad \boldsymbol{\nabla}\mathbf{e}^{\mu} = -\Gamma^{\mu}{}_{\lambda\nu}\mathbf{e}^{\nu}\otimes\mathbf{e}^{\lambda}.$$

These rules, in addition to the usual linearity and Leibniz rule, allow one to construct a gradient of any tensor field, and consequently any kind of derivative within tensor algebra.

Note that the gradient of a vector is a $(1,1)$-tensor, and in general the gradient of a (p,q)-tensor is a $(p,q+1)$-tensor.

Note also that one usually introduces the **connection 1-forms** $\boldsymbol{\omega}^{\lambda}{}_{\mu} \equiv \Gamma^{\lambda}{}_{\mu\nu}\mathbf{e}^{\nu}$ and writes $\boldsymbol{\nabla}\mathbf{e}_{\mu} = \boldsymbol{\omega}^{\lambda}{}_{\mu}\otimes\mathbf{e}_{\lambda}$. Despite their name, $\boldsymbol{\omega}^{\lambda}{}_{\mu}$ are not proper 1-forms — they have indices and consequently depend on the choice of the basis, and their components are connection coefficients, which do not transform as components of a tensor. Nevertheless, we shall see below that connection 1-forms are very useful in practical calculations.

The next topic we are interested in is the operator analogous to $\boldsymbol{\nabla}$, but one adapted to the **algebra of differential forms**, Λ^*. Its explicit construction is similar to the above step-by-step construction of the gradient, and we are not going to deal with it. Instead, like for the gradient, we shall provide simple computational rules which cover the action of this operator on functions and basis 1-forms, and consequently everything else will follow from these rules.

Introduce this operator, called **exterior derivative** and denoted $\boldsymbol{d}$, in the following way:

- Let $\boldsymbol{d}$ act on a p-form to produce a $(p+1)$-form:

$$\boldsymbol{f}_{p+1} = \boldsymbol{d}\boldsymbol{g}_{p}.$$

- Let $\boldsymbol{d}$ be linear:

$$\boldsymbol{d}(a\boldsymbol{f} + b\boldsymbol{g}) = a\boldsymbol{d}\boldsymbol{f} + b\boldsymbol{d}\boldsymbol{g}.$$

- If $\boldsymbol{f}$ and $\boldsymbol{g}$ are p- and q-forms, let the following p-commutative Leibniz rule hold:

$$\boldsymbol{d}(\boldsymbol{f} \wedge \boldsymbol{g}) = \boldsymbol{df} \wedge \boldsymbol{g} + (-1)^p \boldsymbol{f} \wedge \boldsymbol{dg}.$$

- Let $\boldsymbol{d}$ act on a 0-form (ie. an ordinary function) in the same way as a gradient $\boldsymbol{\nabla}$:

$$\boldsymbol{d}f = \boldsymbol{\nabla} f = (\partial_\mu f)\, \mathbf{e}^\mu.$$

- Let double exterior derivative be zero:

$$\boldsymbol{dd} \equiv 0.$$

  This last rule is actually the statement that ordinary partial derivatives should commute, written in the language of differential forms.

- The exterior derivative acts on basis 1-forms according to the equation:

$$\boldsymbol{d}\mathbf{e}^\lambda = -\frac{1}{2} c^\lambda{}_{\mu\nu} \mathbf{e}^\mu \wedge \mathbf{e}^\nu,$$

  where $c^\lambda{}_{\mu\nu}$ are the **commutation coefficients**.

There is one more important remark to be made with respect to commutation coefficients. Note that the exterior derivative (or a gradient) of a 0-form is a 1-form, which can act on a basis vector as a functional, according to the biorthogonality relation:

$$\boldsymbol{d}f[\mathbf{e}_\nu] = \boldsymbol{\nabla} f[\mathbf{e}_\nu] = (\partial_\mu f)\, \mathbf{e}^\mu[\mathbf{e}_\nu] = \partial_\nu f.$$

Now if we remember that we actually introduced the basis vectors $\mathbf{e}_\mu$ as partial differential operators along the coordinate curves,

$$\mathbf{e}_\mu \equiv \frac{\partial}{\partial x^\mu},$$

we can introduce "action" of a vector on a function f as:

$$\mathbf{e}_\mu[f] = \partial_\mu f.$$

Comparing this to the action of $\boldsymbol{d}f$ on $\mathbf{e}_\mu$ we see that $\boldsymbol{d}f[\mathbf{e}_\mu] = \mathbf{e}_\mu[f]$ in the coordinate basis. Choosing f to be precisely the coordinate curve x^ν, we have:

$$\boldsymbol{d}x^\nu[\mathbf{e}_\mu] = \mathbf{e}_\mu[x^\nu] = \partial_\mu x^\nu = \delta^\nu_\mu.$$

But this is nothing else than the biorthogonality relation itself, $\mathbf{e}^\nu[\mathbf{e}_\mu] = \delta^\nu_\mu$, which means that the coordinate basis vectors $\mathbf{e}_\mu \equiv \partial_\mu$ have the following corresponding basis 1-forms:

$$\mathbf{e}^\nu = \boldsymbol{d}x^\nu.$$

Using the rule $\boldsymbol{dd} = 0$ we can now calculate that **in a coordinate basis all commutation coefficients are zero**:

$$c^\lambda{}_{\mu\nu} = 0.$$

The opposite is also valid — **if the commutation coefficients are zero, the basis is a coordinate basis**, ie. there exists a set of coordinate curves x^μ such that

$$\mathbf{e}_\mu = \frac{\partial}{\partial x^\mu} \qquad \text{and} \qquad \mathbf{e}^\mu = \boldsymbol{d}x^\mu.$$

In general, the above action of a basis vector on a function f simply produces another function, on which another basis vector can act. We can therefore introduce the **commutator** of basis vectors as:

$$[\mathbf{e}_\mu, \mathbf{e}_\nu] f = \mathbf{e}_\mu[\mathbf{e}_\nu[f]] - \mathbf{e}_\nu[\mathbf{e}_\mu[f]].$$

One can show that the following identity is satisfied:

$$[\mathbf{e}_\mu, \mathbf{e}_\nu] = c^\lambda{}_{\mu\nu}\mathbf{e}_\lambda$$

Thus, in the coordinate basis $\mathbf{e}_\mu \equiv \partial_\mu$ we see again that commutation coefficients are zero, due to the commutativity of partial derivatives.

Therefore, one can have a convenient criterion to verify whether a given basis is coordinate or non-coordinate — basis is a coordinate basis iff the commutation coefficients are all zero. Given the above formula, those commutation coefficients can be easily calculated.

As a final remark, note that one can extend the idea "basis vector acts on a function" to the idea of "arbitrary vector acts on a function" simply by expanding the vector in a basis:

$$\boldsymbol{A}[f] = A^\mu \mathbf{e}_\mu[f].$$

Consequently, one can define a **commutator of two vectors** as:

$$[\boldsymbol{A}, \boldsymbol{B}]f = \left(A^\mu \mathbf{e}_\mu[B^\lambda] - B^\nu \mathbf{e}_\nu[A^\lambda] + A^\mu B^\nu c^\lambda{}_{\mu\nu}\right)\mathbf{e}_\lambda[f].$$

From this one sees that the commutator of two vectors is a vector. The commutator of vectors is also called **Lie bracket**.

2.3 Curvature and torsion

The directional derivative provides us with information how much a given vector changes with respect to its parallel-transported image in a nearby point in a given direction. Therefore, the statement that the vector $\boldsymbol{A}$ does not actually differ when transported in the direction of vector $\boldsymbol{B}$ is

$$\boldsymbol{\nabla}_{\boldsymbol{B}}\boldsymbol{A} = 0.$$

If this is satisfied, we say that **$\boldsymbol{A}$ stays parallel to itself along $\boldsymbol{B}$**. If a vector stays parallel to itself along its own direction,

$$\boldsymbol{\nabla}_{\boldsymbol{A}}\boldsymbol{A} = 0,$$

we say that the vector **is autoparallel**. If the vector $\boldsymbol{A}$ is a tangent vector to some curve $x^\mu(\lambda)$, namely $\boldsymbol{A} = \frac{dx^\mu}{d\lambda}\mathbf{e}_\mu$, the above autoparallel equation can be rewritten in terms of components as a differential equation for the curve:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu{}_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0.$$

Any curve that satisfies this equation is called an **autoparallel curve**.

Now consider two vectors, $\boldsymbol{A}$ and $\boldsymbol{B}$, and perform first the parallel transport of $\boldsymbol{B}$ along $\boldsymbol{A}$, and then of $\boldsymbol{A}$ along $\boldsymbol{B}$. In flat space we could expect the results to be the same, as the two initial vectors and two transported vectors close up a parallelogram. However, in general this is not always true, and the end-points of transported vectors fail to close. The "missing piece" is a vector that connects the end-points, and is called **torsion** of the manifold. Formally, we define the **torsion operator** and **torsion tensor** as:

$$\mathcal{T}(\boldsymbol{A}, \boldsymbol{B}) = \boldsymbol{\nabla}_{\boldsymbol{A}}\boldsymbol{B} - \boldsymbol{\nabla}_{\boldsymbol{B}}\boldsymbol{A} - [\boldsymbol{A}, \boldsymbol{B}],$$

$$\boldsymbol{T} = \frac{1}{2}T^\lambda{}_{\mu\nu}\mathbf{e}_\lambda \otimes \mathbf{e}^\mu \wedge \mathbf{e}^\nu.$$

The components $T^\lambda{}_{\mu\nu}$ can be expressed in terms of the connection and commutation coefficients as:

$$T^\lambda{}_{\mu\nu} = \Gamma^\lambda{}_{\nu\mu} - \Gamma^\lambda{}_{\mu\nu} - c^\lambda{}_{\mu\nu}.$$

Next consider three vectors, $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$. Construct a closed loop from vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and the "missing" torsion-piece, and perform parallel transport of vector $\boldsymbol{C}$ around this closed loop. In flat space we could expect that the resulting vector should coincide with the original one, but in general this is not the case. The difference is proportional to the **curvature** of the manifold. Formally, we define the **curvature operator** and **curvature tensor** as:

$$\mathcal{R}(\boldsymbol{A},\boldsymbol{B}) = \boldsymbol{\nabla}_{\boldsymbol{A}}\boldsymbol{\nabla}_{\boldsymbol{B}} - \boldsymbol{\nabla}_{\boldsymbol{B}}\boldsymbol{\nabla}_{\boldsymbol{A}} - \boldsymbol{\nabla}_{[\boldsymbol{A},\boldsymbol{B}]},$$

$$\boldsymbol{R} = \frac{1}{2} R^{\lambda}{}_{\rho\mu\nu}\mathbf{e}_{\lambda}\otimes\mathbf{e}^{\rho}\otimes\mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu}.$$

The components $R^{\lambda}{}_{\rho\mu\nu}$ can be expressed in terms of the connection and commutation coefficients as:

$$R^{\lambda}{}_{\rho\mu\nu} = \partial_{\mu}\Gamma^{\lambda}{}_{\rho\nu} + \Gamma^{\lambda}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\rho\nu} - \partial_{\nu}\Gamma^{\lambda}{}_{\rho\mu} - \Gamma^{\lambda}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\rho\mu} - c^{\sigma}{}_{\mu\nu}\Gamma^{\lambda}{}_{\rho\sigma}.$$

Note:

- Torsion and curvature are the properties of the **rule for parallel transport** on a given manifold, and do not depend on vectors $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{C}$ used for their construction. This is obvious when one sees that their components in arbitrary basis depend only on connection and commutation coefficients.
- The dependence on the commutation coefficients can be eliminated by choosing a coordinate basis. However, the dependence on the connection coefficients **cannot be eliminated**.
- If a rule for parallel transport is given such that $\boldsymbol{T} = 0$, we say that the manifold is **torsion-free**.
- If a rule for parallel transport is given such that $\boldsymbol{R} = 0$, we say that the manifold is **flat**. Otherwise it is **curved**.
- It will be useful to rewrite the torsion and curvature tensors in the form

  $$\boldsymbol{T} = \mathbf{e}_{\lambda}\otimes\mathcal{T}^{\lambda}, \qquad \boldsymbol{R} = \mathbf{e}_{\lambda}\otimes\mathbf{e}^{\rho}\otimes\mathcal{R}^{\lambda}{}_{\rho},$$

  where $\mathcal{T}^{\lambda}$ are called **torsion 1-forms** (D of them), while $\mathcal{R}^{\lambda}{}_{\rho}$ are called **curvature 2-forms** (D^2 of them):

  $$\mathcal{T}^{\lambda} \equiv \frac{1}{2}T^{\lambda}{}_{\mu\nu}\mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu}, \qquad \mathcal{R}^{\lambda}{}_{\rho} \equiv \frac{1}{2}R^{\lambda}{}_{\rho\mu\nu}\mathbf{e}^{\mu}\wedge\mathbf{e}^{\nu}.$$

  They are not exactly pure geometric objects (just like connection 1-forms $\boldsymbol{\omega}^{\lambda}{}_{\mu}$), since they carry indices and thus depend on the choice of the basis, but nevertheless they are extremely convenient for calculations, as we shall see below.
- The most efficiently "packaged" statements about the torsion and curvature components are the **Cartan structure equations**:

  $$\mathcal{T}^{\lambda} = \boldsymbol{d}\mathbf{e}^{\lambda} + \boldsymbol{\omega}^{\lambda}{}_{\mu}\wedge\mathbf{e}^{\mu}, \qquad \mathcal{R}^{\lambda}{}_{\rho} = \boldsymbol{d}\boldsymbol{\omega}^{\lambda}{}_{\rho} + \boldsymbol{\omega}^{\lambda}{}_{\mu}\wedge\boldsymbol{\omega}^{\mu}{}_{\rho}.$$

  Later in the text we shall demonstrate how to use these equations to calculate curvature in the most efficient possible way.

Chapter 3

The metric

3.1 Principle of equivalence, the metric tensor

General theory of relativity is based on two important principles — the **principle of general relativity**, which has been mentioned before, and the **equivalence principle**:

> The **equivalence principle** states:
>
> **At every point in an arbitrary gravitational field there exists a reference frame (the so-called locally inertial frame) in which all laws of physics reduce to the form as given in special theory of relativity.**

There are several important things to understand with respect to this axiom:

- One can **always** construct a reference frame which locally "looks" like the flat Minkowski space of special relativity. This is a theorem, and not part of the equivalence principle. We shall deal with this construction below.
- Given the above locally inertial (or locally Minkowskian) frame of reference, it is a matter of **physics** to specify whether all laws look like in special relativity or not. The statement that they do is the nontrivial part of equivalence principle, and it defines the **interaction of gravity with other fields**.

The **Minkowski spacetime** has three very important properties:

- It is **torsion-free and flat**, which means that $\boldsymbol{T} = 0$ and $\boldsymbol{R} = 0$. This in turn means that one can choose a special coordinate basis $\mathbf{e}_i$ where all connection coefficients vanish:

  $$\Gamma^i{}_{jk} = 0.$$

  We use Latin indices instead of Greek to distinguish this special basis from all other arbitrary bases.
- Its **dimension** is $D = 4$.
- It has a concept of **distance**, defined via the **line element formula**

  $$ds^2 = \eta_{ij} dx^i dx^j,$$

  where ds is the distance between the points with coordinates x^i and $x^i + dx^i$ (given in the above special basis $\mathbf{e}_i$), and $\eta_{\mu\nu}$ represent the components of the **metric tensor**:

  $$\boldsymbol{g} \equiv \eta_{ij} \mathbf{e}^i \otimes \mathbf{e}^j = -\mathbf{e}^0 \otimes \mathbf{e}^0 + \mathbf{e}^1 \otimes \mathbf{e}^1 + \mathbf{e}^2 \otimes \mathbf{e}^2 + \mathbf{e}^3 \otimes \mathbf{e}^3.$$

The above equation is the **definition of Minkowski metric**, and we can read off the components η_{ij}:

$$\eta_{00} = -1, \qquad \eta_{11} = \eta_{22} = \eta_{33} = +1, \qquad \eta_{ij} = 0 \text{ for } i \neq j.$$

The concept of distance is **fundamental** to the Minkowski space, and in general.

Of course, in Minkowski space as in any other, one is not required to work in the special basis $\mathbf{e}_i$. One can transform from this basis into another arbitrary one using the transformation matrices $M^\mu{}_i$ and $M^i{}_\mu$ defined earlier. But now we shall introduce another notation for these matrices, $\mathbf{e}^\mu{}_i$ and $\mathbf{e}^i{}_\mu$, and call them **tetrads**. Using the tetrads, one can transform the metric tensor $\boldsymbol{g}$ to an arbitrary basis,

$$\boldsymbol{g} = g_{\mu\nu}\mathbf{e}^\mu \otimes \mathbf{e}^\nu, \qquad g_{\mu\nu} = \eta_{ij}\mathbf{e}^i{}_\mu \mathbf{e}^j{}_\nu, \qquad \eta_{ij} = g_{\mu\nu}\mathbf{e}^\mu{}_i \mathbf{e}^\nu{}_j.$$

Given that $\det[\eta_{\mu\nu}] = -1 \neq 0$ and tetrads are invertible, one can also define the **inverse metric tensor**:

$$\boldsymbol{g}^{-1} = g^{\mu\nu}\mathbf{e}_\mu \otimes \mathbf{e}_\nu, \qquad \boldsymbol{g}\cdot\boldsymbol{g}^{-1} = g_{\mu\lambda}g^{\lambda\nu}\mathbf{e}^\mu \otimes \mathbf{e}_\nu = \delta^\nu_\mu \mathbf{e}^\mu \otimes \mathbf{e}_\nu = \boldsymbol{I},$$

where $\boldsymbol{I}$ is the unit tensor of type $(1,1)$, ie. a unit matrix. The components $g_{\mu\nu}$ and $g^{\mu\nu}$ can thus be regarded as "matrices" inverse to each other.

Given its definition in the special Minkowski coordinate frame $\mathbf{e}_i$, one easily sees that the metric tensor is **symmetric**, and so is its inverse:

$$\boldsymbol{g}^T = \boldsymbol{g}, \qquad \left(\boldsymbol{g}^{-1}\right)^T = \boldsymbol{g}^{-1}, \qquad g_{\mu\nu} = g_{\nu\mu}, \qquad g^{\mu\nu} = g^{\nu\mu}.$$

The metric, being a $(0,2)$-tensor, can be used to define several other concepts:

- Square of the **magnitude of a vector**:

  $$||\mathbf{A}||^2 \equiv \boldsymbol{g}[\mathbf{A},\mathbf{A}] = g_{\mu\nu}A^\rho A^\sigma \mathbf{e}^\mu[\mathbf{e}_\rho] \otimes \mathbf{e}^\nu[\mathbf{e}_\sigma] = g_{\mu\nu}A^\mu A^\nu.$$

  Note that due to the one minus sign in $\eta_{\mu\nu}$ the square of this magnitude can be positive, negative or zero, for nonzero vector $\boldsymbol{A}$. We therefore distinguish between **spacelike, timelike** and **null** (or **lightlike**) vectors, respectively.

- The **scalar product** of two vectors:

  $$\boldsymbol{A}\cdot\boldsymbol{B} \equiv \boldsymbol{g}[\boldsymbol{A},\boldsymbol{B}] = g_{\mu\nu}A^\rho B^\sigma \mathbf{e}^\mu[\mathbf{e}_\rho] \otimes \mathbf{e}^\nu[\mathbf{e}_\sigma] = g_{\mu\nu}A^\mu B^\nu.$$

  Note that this scalar product is not positive-definite. Also note that the square of the magnitude of the vector is actually the scalar product with itself. Finally, note that we use the same "dot" symbol for both the scalar product and inner product. This will be explained in more detail below.

- The scalar product and magnitude of 1-forms is defined in the analogous way, using $\boldsymbol{g}^{-1}$.

- If one chooses some coordinate frame, the basis 1-forms can be written in the form $\mathbf{e}^\mu = \boldsymbol{d}x^\mu$, and one can write the (more rigorous and fancy) **differential geometry line element** as:

  $$\boldsymbol{d}s^2 = \boldsymbol{g} = g_{\mu\nu}\boldsymbol{d}x^\mu \otimes \boldsymbol{d}x^\nu.$$

  One typically omits the $\otimes$ and writes d instead of $\boldsymbol{d}$, thereby recovering $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$. The latter equation is actually just the former one written in sloppy notation.

Once we have introduced the metric, we can generalize to non-Minkowski space. The metric on a general manifold has the same properties as in flat Minkowski case. The only difference is that one has one "copy" of Minkowski space at each point $\mathcal{P}$ on the manifold, and covering only infinitesimal area around the point. This means that at each point one can always choose a basis $\mathbf{e}_i$ where $g_{ij} = \eta_{ij}$. However, in general **this basis is a noncoordinate basis**, which means that one cannot choose a set of coordinate curves which would generate this basis. If this were possible, in these coordinates the space would look flat and with no torsion, which is equivalent to Minkowski space. But given that the basis is noncoordinate, one has nonzero commutation coefficients $c^\lambda{}_{\mu\nu}$ which give rise to nonzero curvature and torsion.

3.2 Associated tensors, index gymnastics

Using the metric tensor and its inverse, we can construct inner products with some other tensor, thereby obtaining a new tensor. This new tensor is called **associated tensor**, and in component notation this operation is called **raising and lowering of indices**. Namely, starting from some tensor $\boldsymbol{A} = A^{\mu\nu}{}_{\lambda}\mathbf{e}_{\mu} \otimes \mathbf{e}_{\nu} \otimes \mathbf{e}^{\lambda}$, we can construct the inner product with the metric:

$$\tilde{\boldsymbol{A}} = \boldsymbol{g} \cdot \boldsymbol{A} = g_{\rho\mu} A^{\mu\nu}{}_{\lambda} \mathbf{e}^{\rho} \otimes \mathbf{e}_{\nu} \otimes \mathbf{e}^{\lambda},$$

where we see that the components of $\tilde{\boldsymbol{A}}$ are $\tilde{A}_{\rho}{}^{\nu}{}_{\lambda} = g_{\rho\mu} A^{\mu\nu}{}_{\lambda}$. The tilde symbol is usually dropped and $\tilde{\boldsymbol{A}}$ is identified with $\boldsymbol{A}$ since they are in one-to-one correspondence with each other. We say that we use the metric tensor to **lower the contravariant index**.

Similarly we can use the inverse metric tensor to **raise the covariant index**, as:

$$\tilde{\boldsymbol{A}} = \boldsymbol{g}^{-1} \cdot \boldsymbol{A} = g^{\rho\lambda} A^{\mu\nu}{}_{\lambda} \mathbf{e}_{\mu} \otimes \mathbf{e}_{\nu} \otimes \mathbf{e}_{\rho},$$

or in components $\tilde{A}^{\mu\nu\rho} = g^{\rho\lambda} A^{\mu\nu}{}_{\lambda}$.

Given that metric tensor is invertible, one can always associate the associated tensor back to the original one, without any loss of information. Thus, through the process of association, the metric tensor enables us to **ignore the distinction between the covariant and contravariant** nature of (p, q)-tensors, and to keep track only of the total tensor rank, $p + q$. This is heavily used in practice, and in combination with other operations enables us to construct new tensors from old.

As we have seen earlier, all operations on tensors can be represented in component language as manipulations with indices. All practical computations with tensors are typically done this way, and informally go by the name **index gymnastics**. Abstract (boldface) language is used to understand the geometric nature of tensors, while any serious nontrivial computation can almost always be performed much more efficiently using component language and index gymnastics.

3.3 Nonmetricity, classification of geometries

Up to now we have introduced two important concepts that characterize the manifold — connection coefficents $\mathit{\Gamma}^{\lambda}{}_{\mu\nu}$ (or equivalently the concept of gradient operator, $\boldsymbol{\nabla}$) and the tetrad coefficients $\mathbf{e}^{\mu}{}_{i}$ (or equivalently the metric tensor, $\boldsymbol{g}$). These two concepts are a priori **completely independent** of each other. However, a lot of information about the differential structure of the manifold can be learned by examining the relation between them. This leads us to several important results and gives new insight into the structure of the geometry of the manifold.

The key relation between metric and connection is embodied in taking the **gradient of the metric**, thereby constructing a new tensor:

$$\boldsymbol{Q} = \boldsymbol{\nabla} \boldsymbol{g}, \qquad \text{or in components} \qquad Q_{\lambda\mu\nu} = \nabla_{\lambda} g_{\mu\nu} \equiv \partial_{\lambda} g_{\mu\nu} - \mathit{\Gamma}^{\sigma}{}_{\mu\lambda} g_{\sigma\nu} - \mathit{\Gamma}^{\sigma}{}_{\nu\lambda} g_{\mu\sigma}.$$

The tensor $\boldsymbol{Q}$ is called **nonmetricity tensor**. The above definition of the nonmetricity tensor can be used together with the first Cartan structure equation $\mathcal{T}^{\lambda} = \boldsymbol{d}\mathbf{e}^{\lambda} + \boldsymbol{\omega}^{\lambda}{}_{\mu} \wedge \mathbf{e}^{\mu}$ and the definition of commutation coefficients $\boldsymbol{d}\mathbf{e}^{\lambda} = -\frac{1}{2} c^{\lambda}{}_{\mu\nu} \mathbf{e}^{\mu} \wedge \mathbf{e}^{\nu}$ to express the connection coefficients in the following form:

$$\mathit{\Gamma}^{\lambda}{}_{\mu\nu} = \left\{ {}^{\lambda}_{\mu\nu} \right\} + \Delta^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + \hat{Q}^{\lambda}{}_{\mu\nu}.$$

Here we have introduced:

- the **Christoffel symbol**

$$\left\{ {}^{\lambda}_{\mu\nu} \right\} \equiv \frac{1}{2} g^{\lambda\sigma} \left(\partial_{\mu} g_{\sigma\nu} + \partial_{\nu} g_{\sigma\mu} - \partial_{\sigma} g_{\mu\nu} \right),$$

- the **Ricci rotation coefficients**

$$\Delta^{\lambda}{}_{\mu\nu} = -\frac{1}{2}\left(c^{\lambda}{}_{\mu\nu} - c_{\nu}{}^{\lambda}{}_{\mu} + c_{\mu\nu}{}^{\lambda}\right),$$

- the **contortion tensor**

$$K^{\lambda}{}_{\mu\nu} = -\frac{1}{2}\left(T^{\lambda}{}_{\mu\nu} - T_{\nu}{}^{\lambda}{}_{\mu} + T_{\mu\nu}{}^{\lambda}\right),$$

- the $\hat{\boldsymbol{Q}}$ tensor

$$\hat{Q}^{\lambda}{}_{\mu\nu} = \frac{1}{2}\left(Q^{\lambda}{}_{\mu\nu} - Q_{\nu}{}^{\lambda}{}_{\mu} - Q_{\mu\nu}{}^{\lambda}\right).$$

The above general formula for connection coefficients has a very interesting structure, which provides us with a means to **classify various geometries** which can be defined on a manifold:

- The general case is just called **linear space with metric**, and denoted $(L_4, \boldsymbol{g})$ in $D = 4$ dimensions. It has in general nonzero nonmetricity, torsion and curvature. The connection coefficients have the form as above,

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\} + \Delta^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu} + \hat{Q}^{\lambda}{}_{\mu\nu}.$$

- The requirement $\boldsymbol{Q} = \boldsymbol{\nabla g} = 0$ (called **metricity condition**) imposes more special structure. The resulting space is called **Riemann-Cartan space**, and denoted U_4 in four dimensions. The connection coefficients have the form

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\} + \Delta^{\lambda}{}_{\mu\nu} + K^{\lambda}{}_{\mu\nu}.$$

- Additional requirement $\boldsymbol{T} = 0$ (**zero torsion condition**) imposes even more special structure. The resulting space is called **Riemann space**, and denoted V_4 in four dimensions. The connection coefficients obtain the form

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\} + \Delta^{\lambda}{}_{\mu\nu}.$$

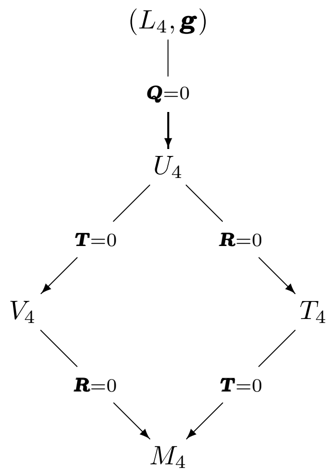
  One can always choose to work in a **coordinate basis**, where the commutation coefficients are zero. In this basis the connection becomes equal to the Christoffel symbol, $\Gamma^{\lambda}{}_{\mu\nu} = \left\{{\lambda \atop \mu\nu}\right\}$, and is called **Levi-Civita connection**.

- Yet another requirement, $\boldsymbol{R} = 0$ (**flat space condition**) specifies the **Minkowski space**, denoted M_4 in four dimensions. In this space one can always choose a special basis such that the connection vanishes:

$$\Gamma^{i}{}_{jk} = 0.$$

- Alternatively, if one starts from the Riemann-Cartan space U_4 and imposes the flat-space condition $\boldsymbol{R} = 0$ while keeping the torsion nonzero, one arrives at the **Weitzenböck's teleparallel space**, denoted T_4. Additional requirement $\boldsymbol{T} = 0$ leads again to Minkowski space M_4.

The above classification of geometries can be neatly represented on a diagram (see below). The general theory of relativity lives in the Riemann space V_4, ie. it assumes that $\boldsymbol{Q} = 0$ and $\boldsymbol{T} = 0$. If one relaxes these conditions, one can construct other, different theories of gravity. The most famous example is the **Einstein-Cartan theory of gravity**, which lives in Riemann-Cartan space U_4.



3.4 Cartan structure equations, calculation of curvature

The general theory of relativity assumes that nonmetricity and torsion are always zero, ie. it is defined in Riemann geometry V_4. The metricity condition can then be rewritten in the form

$$\boldsymbol{d}g_{\mu\nu} = \boldsymbol{\omega}_{\mu\nu} + \boldsymbol{\omega}_{\nu\mu},$$

while the zero-torsion condition can be rewritten (via the first Cartan structure equation) as

$$\boldsymbol{d}\mathbf{e}^{\lambda} + \boldsymbol{\omega}^{\lambda}{}_{\mu} \wedge \mathbf{e}^{\mu} = 0.$$

Starting from the given components of the metric tensor $g_{\mu\nu}$ in some basis, we can use the above two formulas to calculate the connection 1-forms $\boldsymbol{\omega}^{\mu}{}_{\nu}$, and then use the second Cartan structure equation,

$$\mathcal{R}^{\lambda}{}_{\rho} = \boldsymbol{d}\boldsymbol{\omega}^{\lambda}{}_{\rho} + \boldsymbol{\omega}^{\lambda}{}_{\mu} \wedge \boldsymbol{\omega}^{\mu}{}_{\rho},$$

to calculate curvature tensor.

The general procedure goes as follows:

- Use convenient tetrads $\mathbf{e}^{\mu}{}_{i}$ and $\mathbf{e}^{i}{}_{\mu}$ to transform into a locally inertial frame $\mathbf{e}_i$, so that the components of the metric tensor are $g_{ij} = \eta_{ij}$. From the metricity condition then conclude that $\boldsymbol{\omega}_{ij} = -\boldsymbol{\omega}_{ji}$, ie. there are only 6 connection 1-forms to be calculated (in $D = 4$ dimensions).
- Calculate the commutation coefficients in the inertial basis from their definition,

$$\boldsymbol{d}\mathbf{e}^{i} = -\frac{1}{2} c^{i}{}_{jk} \mathbf{e}^{j} \wedge \mathbf{e}^{k}.$$

- Employ the first Cartan structure equation,

$$\boldsymbol{d}\mathbf{e}^{i} + \boldsymbol{\omega}^{i}{}_{j} \wedge \mathbf{e}^{j} = 0,$$

  to express the connection 1-forms in terms of commutation coefficients. This step has already been done above, in the general expression for the connection in Riemann space:

$$\Gamma^{\lambda}{}_{\mu\nu} = \left\{ {\lambda \atop \mu\nu} \right\} + \Delta^{\lambda}{}_{\mu\nu}.$$

  The Christoffel symbol is equal to zero in locally inertial frame because $g_{ij} = \eta_{ij} = const$, so we have

$$\boldsymbol{\omega}_{ij} = -\frac{1}{2} \left(c_{ijk} - c_{kij} + c_{jki} \right) \mathbf{e}^{k}.$$

- Employ the second Cartan structure equation,

$$\mathcal{R}^i{}_j = \boldsymbol{d}\boldsymbol{\omega}^i{}_j + \boldsymbol{\omega}^i{}_k \wedge \boldsymbol{\omega}^k{}_j,$$

  to calculate the components of the curvature 2-forms $\mathcal{R}^i{}_j$, and tabulate the nonzero components of the Riemann tensor,

$$\mathcal{R}^i{}_j \equiv \frac{1}{2} R^i{}_{jkl} \mathbf{e}^k \wedge \mathbf{e}^l.$$

- Finally, recover the components of the Riemann tensor in the original coordinates by transforming back from the locally inertial ones:

$$R^\lambda{}_{\rho\mu\nu} = \mathbf{e}^\lambda{}_i \mathbf{e}^j{}_\rho \mathbf{e}^k{}_\mu \mathbf{e}^l{}_\nu R^i{}_{jkl}.$$

  This completes the calculation of the curvature tensor from the given metric.

We now demonstrate this procedure for the well-known case of the Friedmann metric. This metric is usually written using the natural spherical coordinates of a 3-sphere, χ, θ, φ, and time t, as:

$$ds^2 = -dt^2 + a^2(t)d\chi^2 + a^2(t)\sin^2\chi d\theta^2 + a^2(t)\sin^2\chi\sin^2\theta d\varphi^2,$$

or in more rigorous and formal notation,

$$\boldsymbol{d}s^2 \equiv \boldsymbol{g} = -\boldsymbol{d}t \otimes \boldsymbol{d}t + a^2(t)\boldsymbol{d}\chi \otimes \boldsymbol{d}\chi + a^2(t)\sin^2\chi \boldsymbol{d}\theta \otimes \boldsymbol{d}\theta + a^2(t)\sin^2\chi\sin^2\theta \boldsymbol{d}\varphi \otimes \boldsymbol{d}\varphi.$$

The metric is represented in the natural coordinate basis $\mathbf{e}^\mu = \boldsymbol{d}x^\mu$, ie.

$$\mathbf{e}_t = \boldsymbol{d}t, \qquad \mathbf{e}_\chi = \boldsymbol{d}\chi, \qquad \mathbf{e}_\theta = \boldsymbol{d}\theta, \qquad \mathbf{e}_\varphi = \boldsymbol{d}\varphi,$$

and its components $g_{\mu\nu}$ are

$$g_{tt} = -1, \qquad g_{\chi\chi} = a^2(t), \qquad g_{\theta\theta} = a^2(t)\sin^2\chi, \qquad g_{\varphi\varphi} = a^2(t)\sin^2\chi\sin^2\theta.$$

Now we transform into the locally inertial basis, $\mathbf{e}^i = \mathbf{e}^i{}_\mu \mathbf{e}^\mu$, such that

$$\boldsymbol{d}s^2 \equiv \boldsymbol{g} = -\mathbf{e}^0 \otimes \mathbf{e}^0 + \mathbf{e}^1 \otimes \mathbf{e}^1 + \mathbf{e}^2 \otimes \mathbf{e}^2 + \mathbf{e}^3 \otimes \mathbf{e}^3,$$

ie. such that the components of the metric tensor are $g_{ij} = \eta_{ij}$. The tetrads $\mathbf{e}^i{}_\mu$ that perform this are obvious:

$$\mathbf{e}^0 = \underbrace{1}_{\mathbf{e}^0{}_t}\boldsymbol{d}t, \qquad \mathbf{e}^1 = \underbrace{a}_{\mathbf{e}^1{}_\chi}\boldsymbol{d}\chi, \qquad \mathbf{e}^2 = \underbrace{a\sin\chi}_{\mathbf{e}^2{}_\theta}\boldsymbol{d}\theta, \qquad \mathbf{e}^3 = \underbrace{a\sin\chi\sin\theta}_{\mathbf{e}^3{}_\varphi}\boldsymbol{d}\varphi.$$

First we calculate the commutation coefficients. Differentiating $\mathbf{e}^0$ we have:

$$\boldsymbol{d}\mathbf{e}^0 = \boldsymbol{d}\boldsymbol{d}t \equiv 0.$$

Differentiating $\mathbf{e}^1$ we have:

$$\boldsymbol{d}\mathbf{e}^1 = \boldsymbol{d}\left(a(t)\boldsymbol{d}\chi\right) = (\boldsymbol{d}a) \wedge \boldsymbol{d}\chi + a\underbrace{\boldsymbol{d}\boldsymbol{d}\chi}_{0} = \dot{a}\boldsymbol{d}t \wedge \boldsymbol{d}\chi = \frac{\dot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^1 = -c^1{}_{01}\mathbf{e}^0 \wedge \mathbf{e}^1.$$

Similarly, we have

$$\boldsymbol{d}\mathbf{e}^2 = \frac{\dot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^2 + \frac{1}{a}\cot\chi \mathbf{e}^1 \wedge \mathbf{e}^2,$$

and

$$\boldsymbol{d}\mathbf{e}^3 = \frac{\dot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^3 + \frac{1}{a}\cot\chi \mathbf{e}^1 \wedge \mathbf{e}^3 + \frac{\cot\theta}{a\sin\chi}\mathbf{e}^2 \wedge \mathbf{e}^3.$$

From this we read off all nonzero commutation coefficients (indices $0, 1, 2, 3$ are of course raised and lowered using $\eta_{\mu\nu}$):

$$c_{101} = c_{202} = c_{303} = -\frac{\dot{a}}{a}, \qquad c_{212} = c_{313} = -\frac{1}{a}\cot\chi, \qquad c_{323} = -\frac{\cot\theta}{a\sin\chi}.$$

Note that due to antisymmetry $c_{ijk} = c_{ikj}$ there are also other nonzero coefficients that can be obtained from these.

Next we use these coefficients to construct 6 connection 1-forms (using the formula given above). Straightforward (and short!) computation gives:

$$\begin{aligned}
\boldsymbol{\omega}^0{}_1 &= \frac{\dot{a}}{a}\mathbf{e}^1 = \dot{a}\boldsymbol{d}\chi, & \boldsymbol{\omega}^1{}_2 &= -\frac{1}{a}\cot\chi = -\cos\chi\boldsymbol{d}\theta, \\
\boldsymbol{\omega}^0{}_2 &= \frac{\dot{a}}{a}\mathbf{e}^2 = \dot{a}\sin\chi\boldsymbol{d}\theta, & \boldsymbol{\omega}^1{}_3 &= -\frac{1}{a}\cot\chi = -\cos\chi\sin\theta\boldsymbol{d}\varphi, \\
\boldsymbol{\omega}^0{}_3 &= \frac{\dot{a}}{a}\mathbf{e}^3 = \dot{a}\sin\chi\sin\theta\boldsymbol{d}\varphi, & \boldsymbol{\omega}^2{}_3 &= -\frac{\cot\theta}{a\sin\chi} = -\cos\theta\boldsymbol{d}\varphi.
\end{aligned}$$

The connection 1-forms are properly expressed with components in the basis $\mathbf{e}^i$, but we have also expressed them in the coordinate basis $\mathbf{e}^\mu = \boldsymbol{d}x^\mu$ since this is more convenient for the next step.

The next step involves calculating 6 curvature 2-forms, using the second Cartan structure equation. For example,

$$\mathcal{R}^0{}_1 = \boldsymbol{d}\boldsymbol{\omega}^0{}_1 + \boldsymbol{\omega}^0{}_2 \wedge \boldsymbol{\omega}^2{}_1 + \boldsymbol{\omega}^0{}_3 \wedge \boldsymbol{\omega}^3{}_1 = \ddot{a}\boldsymbol{d}t \wedge \boldsymbol{d}\chi + 0 + 0 = \frac{\ddot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^1 = R^0{}_{101}\mathbf{e}^0 \wedge \mathbf{e}^1.$$

In a similar way we compute all curvature 2-forms:

$$\mathcal{R}^0{}_1 = \frac{\ddot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^1, \qquad \mathcal{R}^0{}_2 = \frac{\ddot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^2, \qquad \mathcal{R}^0{}_3 = \frac{\ddot{a}}{a}\mathbf{e}^0 \wedge \mathbf{e}^3,$$

$$\mathcal{R}^1{}_2 = \frac{1+\dot{a}^2}{a}\mathbf{e}^1 \wedge \mathbf{e}^2, \qquad \mathcal{R}^1{}_3 = \frac{1+\dot{a}^2}{a}\mathbf{e}^1 \wedge \mathbf{e}^3, \qquad \mathcal{R}^2{}_3 = \frac{1+\dot{a}^2}{a}\mathbf{e}^2 \wedge \mathbf{e}^3.$$

From here we simply read off the nonzero components of the curvature tensor in the locally inertial frame:

$$R^0{}_{101} = R^0{}_{202} = R^0{}_{303} = \frac{\ddot{a}}{a}, \qquad R^1{}_{212} = R^1{}_{313} = R^2{}_{323} = \frac{1+\dot{a}^2}{a}.$$

Finally, we use the tetrads to convert these components back to the original coordinate frame, to obtain the final result:

$$\begin{aligned}
R^t{}_{\chi t\chi} &= \ddot{a}a, & R^\chi{}_{\theta\chi\theta} &= (1+\dot{a}^2)\sin^2\chi, \\
R^t{}_{\theta t\theta} &= \ddot{a}a\sin^2\chi, & R^\chi{}_{\varphi\chi\varphi} &= (1+\dot{a}^2)\sin^2\chi\sin^2\theta, \\
R^t{}_{\varphi t\varphi} &= \ddot{a}a\sin^2\chi\sin^2\theta, & R^\theta{}_{\varphi\theta\varphi} &= (1+\dot{a}^2)\sin^2\chi\sin^2\theta.
\end{aligned}$$

All other components are obtained from these using various symmetries of the curvature tensor.

This completes the calculation of curvature. Once curvature tensor has been computed, it is straightforward and easy to construct the components of Ricci tensor, scalar curvature and Einstein tensor, using equations

$$R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu}, \qquad R = g^{\mu\nu}R_{\mu\nu}, \qquad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R.$$

Happy calculating! ;-)

For further reading

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[2] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, “Gravitation”, W. H. Freeman and Co. (1973), ISBN 978-0-7167-0344-0.