STRING DYNAMICS IN CURVED SPACETIMES

M. Vasilić, M. Vojinović

Phys. Rev. D **73**, 124013 (2006).

MOTIVATION

$$S = T \int_{\mathcal{M}_2} d^2\xi \, \sqrt{-h} \left[\left(\frac{1}{2} h^{ab} g_{\mu\nu}(x) + \frac{\varepsilon^{ab}}{\sqrt{-h}} B_{\mu\nu}(x) \right) \frac{\partial z^\mu}{\partial \xi^a} \frac{\partial z^\nu}{\partial \xi^b} + \Phi(x) R^{(2)} \right]$$

where

$$B_{\mu\nu} \leftrightarrow T^{\mu}{}_{\nu\lambda}, \qquad \Phi \leftrightarrow Q^{\mu}{}_{\nu\lambda}$$

The systematic treatment of motion of a string in spacetime with background curvature and torsion is required!

PROBLEM

Derive equations of motion of the string in spacetime with background curvature and torsion.

Complicated!!! Examine the torsionless case first.

METHOD

Generalization of Papapetrou analysis from particles to extended objects

[Einstein, Infeld, Born, Fock, Mathisson, . . . , Papapetrou. Torsion generalizations: Yasskin and Stoeger (1980), Nomura, Hayashi, Shirafuji (1991).]

SOME MATH

$$f(x) = \sum_{k\in\mathbb{N}_0} b_k \frac{d^k}{dx^k}\delta(x) = b_0\delta(x) + b_1\frac{d}{dx}\delta(x) + b_2\frac{d^2}{dx^2}\delta(x) + \ldots$$

$$b_k = \frac{(-1)^k}{k!}\int dx\ x^k f(x), \qquad b_0 = \int dx\ f(x), \quad b_1 = -\int dx\ xf(x), \quad \ldots$$

$f(x)$ must be decaying faster than any power of x, ie. it must decay exponentially, or faster.

OUTLINE OF THE METHOD

Start with the field theory Lagrangian:

$$\mathcal{L} = -\frac{1}{16\pi G}\sqrt{-g}R + \mathcal{L}_M(\phi, \partial\phi, g, \Gamma, R, \ldots)$$

Equations of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G\, T_{\mu\nu}, \qquad \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} - \frac{\partial\mathcal{L}}{\partial\phi} = 0$$

Important consequences:

$$\nabla_\nu T^{\mu\nu} = 0, \qquad T^{\mu\nu} = T^{\nu\mu}$$

Description of matter: choose surface $\mathcal{M}$ ($\dim\mathcal{M} = p+1 < D$), as $x^\mu = z^\mu(\xi^a)$, and expand the stress–energy tensor as:

$$T^{\mu\nu} = \int d^{p+1}\xi \, \sqrt{-\gamma} \left[b^{\mu\nu}\frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + b^{\mu\nu\rho}\nabla_\rho \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + b^{\mu\nu\rho\sigma}\nabla_\sigma\nabla_\rho \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + \dots \right]$$

Basic assumption:

Matter equations of motion have a "stringlike kink solution", ie. matter is localized along some line, while curvature is not.

$$\Updownarrow$$

$$b^{\mu\nu} \gg b^{\mu\nu\rho} \gg b^{\mu\nu\rho\sigma} \gg \dots$$

Example: Nielsen-Olesen flux tube solution in Higgs type scalar electrodynamics

$$\mathcal{L}_M = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + (D_\mu\phi)^*(D^\mu\phi) - \lambda\left(\phi^*\phi - a^2\right)^2$$

Single-pole approximation for stress–energy tensor:

$$T^{\mu\nu} = \int d^{p+1}\xi \, \sqrt{-\gamma} \, b^{\mu\nu}\frac{\delta^{(D)}(x-z)}{\sqrt{-g}}$$

Sketchy calculation — employ equations $\nabla_\nu T^{\mu\nu} = 0$ and $T^{\mu\nu} = T^{\nu\mu}$ to obtain:

$$\int d\xi \left[\text{TermI}\right] \delta(x-z) + \left[\text{TermII}\right] \partial\delta(x-z) + \partial \left[\text{BoundaryTerm}\right] = 0$$

$$\text{TermI} = 0 \quad \wedge \quad \text{TermII} = 0 \qquad \Rightarrow \qquad b^{\mu\nu} = m^{ab} u^\mu_a u^\nu_b, \qquad \nabla_a (m^{ab} u^\mu_b) = 0$$

$$\text{BoundaryTerm} = 0 \qquad \Rightarrow \qquad \sqrt{-\gamma} m^{ab} u^\mu_a n_b|_{\partial\mathcal{M}} = 0$$

Compare with Nambu-Goto equations of motion and Neumann boundary conditions

$$\nabla_a (\gamma^{ab} u^\mu_b) = 0, \qquad \sqrt{-\gamma}\gamma^{ab} u^\mu_a n_b|_{\partial\mathcal{M}} = 0$$

Also compare the stress–energy tensor:

$$T^{\mu\nu}_{\text{NG}} = T \int d^{p+1}\xi \, \sqrt{-\gamma}\, \gamma^{ab} u^\mu_a u^\nu_b \frac{\delta^{(D)}(x-z)}{\sqrt{-g}}$$

$$T^{\mu\nu} = \int d^{p+1}\xi \, \sqrt{-\gamma}\, m^{ab} u^\mu_a u^\nu_b \frac{\delta^{(D)}(x-z)}{\sqrt{-g}}$$

and conclude:

$$\boxed{m^{ab}_{\text{NG}} = T\gamma^{ab}}$$

INTERPRETATION

From equations of motion $\nabla_a(m^{ab}u_b^\mu) = 0$ **deduce that**

$$\nabla_a m^{ab} = 0, \qquad m^{ab} = m^{ba}$$

and interpret the mass tensor m^{ab} **as the effective 2-dimensional stress–energy tensor! Use the expreession for** $T^{\mu\nu}$ **to read off it's components:**

$$\left[m^{ab}\right] = \begin{bmatrix} \rho & \pi \\ \pi & p \end{bmatrix} \qquad \textbf{in a suitable frame}$$

Canonical cases:

- **Massive string:**

$$\left[m^{ab}\right] = \begin{bmatrix} \lambda^{(1)} & 0 \\ 0 & -\lambda^{(2)} \end{bmatrix}, \qquad |v| < 1;$$

- **Massless string:**

$$\left[m^{ab}\right] = \begin{bmatrix} \mu & \mu \\ \mu & \mu \end{bmatrix}, \qquad |v| = 1;$$

- **Nambu-Goto string:**

$$\left[m^{ab}\right] = \begin{bmatrix} T & 0 \\ 0 & -T \end{bmatrix}, \qquad \begin{array}{r} |v| < 1, \\ |v|_{\partial\mathcal{M}} = 1; \end{array}$$

- **Tachyonic string:**

$$\left[m^{ab}\right] = \begin{bmatrix} \lambda' & \lambda'' \\ \lambda'' & -\lambda' \end{bmatrix}, \qquad |v| > 1.$$

THE RESULT

$m^{ab}\nabla_a u_b^\mu = 0$ **is a GENERAL equation of motion for ANY stringlike matter in curved spacetime, where the string is made of matter fields of**

the usual CLASSICAL FIELD THEORY !!!

This is a good framework for exploring motion of stringlike matter, because torsion can be naturally incorporated.

Also, we obtain general Neumann-like boundary conditions *automatically!*

BEYOND THE RESULT

Include more moments — pole-dipole approximation:

$$T^{\mu\nu} = \int d^{p+1}\xi \, \sqrt{-\gamma} \left[b^{\mu\nu} \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} + b^{\mu\nu\rho} \nabla_\rho \frac{\delta^{(D)}(x-z)}{\sqrt{-g}} \right]$$

Repeat the whole procedure to end up with

- **equations of motion for the string:**

$$\nabla_c \left[m^{ac} u_a^\mu + 2 u_\sigma^c \nabla_a S_\perp^{[\mu\sigma]a} - u_b^\mu u_\rho^b u_\sigma^c \nabla_a S_\perp^{[\rho\sigma]a} \right] + u_a^\lambda S_\perp^{[\nu\rho]a} R^\mu{}_{\lambda\nu\rho} = 0.$$

- **equation for string angular momentum:**

$$\nabla_a S_\perp^{[\mu\lambda]a} - u_b^\mu u_\nu^b \nabla_a S_\perp^{[\nu\lambda]a} - u_b^\lambda u_\sigma^b \nabla_a S_\perp^{[\mu\sigma]a} + u_b^\mu u_\nu^b u_c^\lambda u_\sigma^c \nabla_a S_\perp^{[\nu\sigma]a} = 0.$$

- **two sets of boundary conditions:**

$$\sqrt{-\gamma} n_c \left(m^{ac} u_a^\mu + 2 u_\sigma^c \nabla_a S_\perp^{[\mu\sigma]a} - u_b^\mu u_\rho^b u_\sigma^c \nabla_a S_\perp^{[\rho\sigma]a} \right) \Big|_{\partial\mathcal{M}} = 0, \qquad \sqrt{-\gamma} n_c S_\perp^{[\mu\sigma]c} \Big|_{\partial\mathcal{M}} = 0.$$

[Analysis yet to be completed...]

RESEARCH DIRECTIONS

- **Strings connected to p-branes**
- **Strings with massive particles on the boundary**
- **Interaction of strings**
- **Etc. . .**